A discrete representation of free MV-algebras

Antonio Di Nola[1*], **Revaz Grigolia**[2**], and **Luca Spada**[1***†]

[1] Department of Mathematics and Computer Sciences. University of Salerno. Via ponte don Melillo, 84084 Fisciano (SA), Italy
[2] Institute of Computer Sciences. Tbilisi State University. Sandro Euli str., 5, 0186 Tbilisi, Georgia

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We prove that the n-generated free MV-algebra is isomorphic to a quotient of the disjoint union of all the n-generated free MV$^{(n)}$-algebras. Such a quotient can be seen as the direct limit of a system consisting of all free MV$^{(n)}$-algebras and special maps between them as morphisms.

The variety of MV-algebras is the equivalent algebraic semantics of the infinitely-valued Łukasiewicz logic [1], and MV$_n$-algebras are algebraic models of the Łukasiewicz logic with n truth values ($2 \leq n < \omega$).

Recall that an algebra $A \in \mathbb{V}$ is said to be a *free algebra* in a variety $\mathbb{V}$, if there exists a set $A_0 \subset A$ such that A_0 generates A and every mapping f from A_0 to any algebra $B \in \mathbb{V}$ can be extended to a homomorphism h from A to B. In this case A_0 is said to be *the set of free generators* of A. If the set of free generators has cardinality $m \in \omega$ then A is said *m-generated* or, in general, *finitely generated*.

The structure of non-equivalent formulas of Łukasiewicz propositional logic forms the free ω-generated MV-algebra, through the well-known *Tarki-Lindenbaum* construction. If we restrict to non-equivalent formulas with m propositional variables, then we obtain the m-generated free MV-algebra.

R. McNaughton described a set of special functions $f : [0,1]^m \longrightarrow [0,1]$, given by all the continuous piecewise linear functions with integer derivatives, called after him *McNaughton functions*. Such a set, when endowed with MV-operations defined component-wise, is isomorphic to the m-generated free MV-algebra [6]. It is worth to stress that the characterisation due to McNaughton was improved in [7, 8]. Another characterisation of the free MV-algebrea is given in [9]. Let $\mathbb{MV}_n$ indicate the subvariety of $\mathbb{MV}$ (the variety of all MV-algebras), given by all MV$_n$-algebras. In [3, 4] is given the description of the m-generated free MV-algebra as a subalgebra of the inverse limit of a system consisting of m-generated free MV-algebras in the variety $\mathbb{MV}_n$. A closely related family of varieties is given by $\mathbb{MV}^{(n)}$ for $n \in \omega$, where each of them sits between $\mathbb{MV}$ and $\mathbb{MV}_n$.

An algebra is *locally finite* if all its finitely generated subalgebras are finite. Recall also that a variety is called *locally finite* if all its finitely generated members are finite. The link between the two concepts is given by the fact that a variety is locally finite if, and only if, its free algebra over ω generators is locally finite. It is known that the variety $\mathbb{MV}$ is not locally finite but, remarkably, it is generated by all simple finite MV-algebras. In addition we have that the subvarieties of $\mathbb{MV}$ which are generated by finite families of simple finite (finite and linearly ordered) MV-algebras are locally finite. So finitely generated free algebras in any variety $\mathbb{MV}_n$, as well as $\mathbb{MV}^{(n)}$, are finite, while finite generated free algebras in $\mathbb{MV}$ are infinite.

For any integer $m \geq 1$, we give a representation of the free m-generated MV-algebra in $\mathbb{MV}$, denoted by $F_{\mathbb{MV}}(m)$, using in a suitable manner, all the free m-generated algebras from $\mathbb{MV}^{(n)}$, for every n, denoted by $F_{\mathbb{MV}^{(n)}}(m)$. A similar approach was studied in [2], although the authors only deal with the case with one generator.

* email: adinola@unisa.it
** email: grigolia@yahoo.com
*** email: lspada@unisa.it
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Unfortunately it is impossible to define a family of morphisms which send in general $F_{\mathbb{MV}^{(n)}}(m)$ in $F_{\mathbb{MV}^{(n+1)}}(m)$. For this reason, in order to build the desired direct limit, we will forget all the structure of MV-algebras, and we will give a canonical method to re-endow the direct limit with the necessary structure of MV-algebra. The natural question whether it is possible to uniformly define a family of *lattice*-embeddings between $F_{\mathbb{MV}^{(n)}}(m)$ and $F_{\mathbb{MV}^{(n+1)}}(m)$, for any $n \in \omega$ remains unanswered to us.

We will use the case with one generator to explore concretely the construction of the direct limit and we will use it as a springboard to abstract and generalise to the case with more than one generator.

1 Preliminaries

We recall that an algebra $A = (A; \oplus, \neg, 0)$ is said to be an MV-algebra, [1], if it satisfies the following equations:

(i) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$; (ii) $x \oplus y = y \oplus x$;

(iii) $x \oplus 0 = x$; (iv) $x \oplus \neg 0 = \neg 0$;

(v) $\neg\neg x = x$ (vi) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Notation 1.1 It is often useful to consider also the following defined operations.

$$1 := \neg 0; \qquad x \odot y := \neg(\neg x \oplus \neg y);$$

We shall also drop the $\odot$ symbol, writing ab for $a \odot b$ and use the following abbreviations:

$$a^n = \underbrace{a \odot \cdots \odot a}_{n \text{ times}} \quad \text{and} \quad (n)a = \underbrace{a \oplus \cdots \oplus a}_{n \text{ times}}.$$

Every MV-algebra has an underlying ordered structure defined by $x \leq y$ iff $\neg x \oplus y = 1$ and $(A; \leq, 0, 1)$ is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra:

$$xy \leq x \wedge y \leq x \vee y \leq x \oplus y.$$

The unit interval of real numbers $[0, 1]$ endowed with the following operations,

$$x \oplus y = \min(1, x + y) \quad \text{and} \quad \neg x = 1 - x,$$

is an MV-algebra. It is well known that the MV-algebra $S = ([0, 1], \oplus, \neg, 0)$ generates the variety $\mathbb{MV}$, in symbols $\mathcal{V}(S) = \mathbb{MV}$.

The subvariety $\mathbb{MV}^{(n)} \subset \mathbb{MV}$ is axiomatized by the extra axiom: $(n + 1)x = (n)x$.
Let us write ω_0 for $\omega \setminus \{0\}$; for $n \in \omega_0$ we set $S_n = (S_n; \oplus, \neg, 0)$, where

$$S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$$

and the operations $\oplus, \neg$ are defined as in S. Then $\mathbb{MV}^{(n)} = \mathcal{V}(\{S_1, ..., S_n\})$.

Notation 1.2 Since the context does not allow confusion we will write $\mathcal{F}^{(n)}(m)$ for $F_{\mathbb{MV}^{(n)}}(m)$ and $\mathcal{F}^{(n)}$ for $F_{\mathbb{MV}^{(n)}}(1)$. Furthermore $\mathcal{F}(m)$ will be used instead of $F_{\mathbb{MV}}(m)$ and simply $\mathcal{F}$ for $F_{\mathbb{MV}}(1)$

The description of $\mathcal{F}^{(n)}(m)$ can be found in [4], we recall some of the results contained therein.

Definition 1.3 The function $v_m(x)$ is defined as follows:

$$v_m(1) = 2^m,$$
$$v_m(2) = 3^m - 2^m,$$
$$\vdots$$
$$v_m(n) = (n+1)^m - (v_m(n_1) + ... + v_m(n_{k-1})),$$

where $n_1 = 1, n_k = n$ and $n_2, ..., n_{k-1}$ are all the strict divisors of n.

The following characterisation will be heavily used in the paper:

Theorem 1.4 ([3, Lemma 2.2]) *With the above notation:*

$$\mathcal{F}^{(n)}(m) \cong S_1^{v_m(1)} \times ... \times S_n^{v_m(n)}.$$

Proposition 1.5 ([3]) *Given a tuple $(a_1, ..., a_k)$ in $\mathcal{F}^{(n)}(m)$ there is a $(m+1)$-ary McNaughton function f such that the set $\{a_1, ..., a_k\}$ is exactly the range of f restricted to $\bigcup_{i=1}^{n} S_i$.*

2 Representation of 1-generated free MV-algebras

The above theorem gives a geometrical interpretation of the algebras $\mathcal{F}^{(n)}(m)$, indeed, modulo a permutation of the components, we can think of its elements as point in the $(m+1)$-space.

Figure 1 represents an element of $\mathcal{F}^{(5)}(1)$ in two ways. In the first part the component are ordered following the natural order $S_1, S_1, S_2, S_3, S_3...$ in the second part the components are disposed in order to match their values in the square $[0,1] \times [0,1]$.

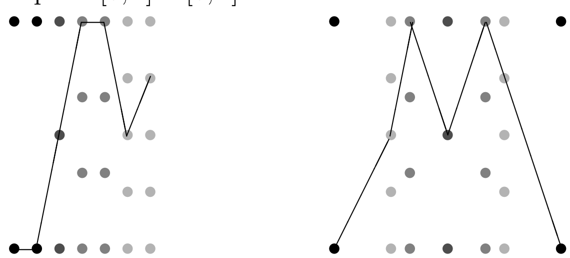

Fig. 1 The black lines in the figures depict the same element of $\mathcal{F}^{(5)}(1)(= S_1^2 \times S_2 \times S_3^2 \times S_4^2 \times S_5^4)$

Both representations will be suggestive in the following; next definition enables us to formalise the two representations of Figure 1.

Definition 2.1 $\mathcal{Q}$ is the set of irreducible fractions between 0 and 1, endowed with the natural order, which we will indicate as usual with $<$.
$\mathcal{Q}^{\prec}$ has the same domain of $\mathcal{Q}$ but its linear order $\prec$ is given by

$$\frac{m}{n} \prec \frac{p}{q} \text{ if, and only if, } n < q \text{ or, if } n = q \text{ then } m < p$$

So the $\prec$-sorted listing of $\mathcal{Q}$ is $\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, ...\}$.

The $<$-ordered version of $\mathcal{Q}$ will give the indexes of the element of the free algebras as in the right part of Figure 1, while the $\prec$-sorted version of $\mathcal{Q}$ will serve for the left-hand. In the latter case, if for some $k \in \omega_0$ the tuple $(a_1, a_2, ..., a_k)$ is an arbitrary element of $\mathcal{F}^{(n)}(m)$, we will equivalently denote it by $(a_{\frac{0}{1}}, a_{\frac{1}{1}}, a_{\frac{1}{2}}, ..., a_q)$, where q is the k^{th} element of $\mathcal{Q}^{\prec}$.

Thanks to this notation we will think of a tuple $(a_{\frac{0}{1}}, a_{\frac{1}{1}}, a_{\frac{1}{2}}, ..., a_q)$ as a sequence of points in the real plane given by the pairs

$$(\frac{0}{1}, a_{\frac{0}{1}}), (\frac{1}{1}, a_{\frac{1}{1}}), (\frac{1}{2}, a_{\frac{1}{2}}), ..., (q, a_q),$$

for any q in $\mathcal{Q}^{\prec}$.

The notation may look a little bit unconventional but it will allow to express in a succinct way the constructions carried out in the rest of the paper.

Notation 2.2 Following the above discussion when we write $(a_q)_{q \in \mathcal{Q}^{\prec}}$ we denote the ordered sequence $(a_{\frac{0}{1}}, a_{\frac{1}{1}}, ...)$, just in the same way as $(a_n)_{n \in \mathbb{N}}$ usually denotes the ordered sequence $a_1, a_2, ...$

The object in the direct system we want to construct will be the set-reduct (which we still indicate with the same symbol) of all MV-algebras $\mathcal{F}^{(n)}$. Next, we define a family of embeddings $\varepsilon_k : \mathcal{F}^{(k)} \to \mathcal{F}^{(k+1)}$.

The intuitive idea behind the definition of ε is as follows. Given a tuple $(a_1, ..., a_n)$ in $\mathcal{F}^{(k)}$ we know (Theorem 1.4) that there is a McNaughton function $f(x)$ such that the set $\{a_1, ..., a_n\}$ is exactly the range of $f(x)$ restricted to $\bigcup_{i=1}^{k} S_k$. Hence we define $\varepsilon(a_1, ..., a_n)$ as the tuple given by the domain of $f(x)$ when restricted to $\bigcup_{i=1}^{k+1} S_{k+1}$.

Of course, there can be several functions satisfying the hypothesis, but we provide a procedure which forces us to select the "simplest" one (i.e. the one with the minimum number of linear pieces).

More precisely, if we index the tuple $(a_1, ..., a_n)$ by an initial segment of $\mathcal{Q}$, $(a_{\frac{0}{1}}, a_{\frac{1}{1}}, ..., a_{\frac{k-1}{k}})$, we define the new components $a_{\frac{1}{k+1}}, ..., a_{\frac{k}{k+1}}$ by interpolation. So if we want to find the value of an arbitrary $a_{\frac{i}{k+1}}$ we shall look for the closest fractions above and below $\frac{i}{k+1}$, say $\frac{m}{n}$ and $\frac{p}{q}$ such that $n, q < k+1$. Then we calculate the function of the line which connects $(\frac{m}{n}, a_{\frac{m}{n}})$ and $(\frac{p}{q}, a_{\frac{p}{q}})$ and finally we evaluate such a function on the point $\frac{i}{k+1}$.

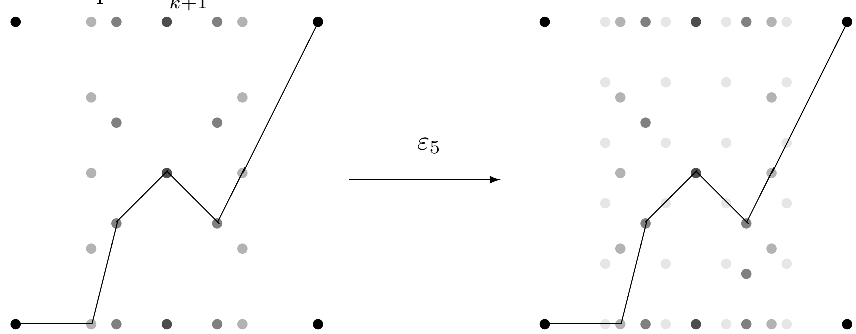

Fig. 2 ε_5 sending an element of $\mathcal{F}_5$ to an element of $\mathcal{F}_6$.

We now formalise this idea. To begin with we define the two functions which, given a fraction $\frac{m}{n} \in \mathcal{Q}$, give the smallest and the greatest fractions in $\mathcal{Q}$ which are above and, respectively, below $\frac{m}{n}$ and have denominators smaller then n.

Definition 2.3 Let us define for any $\frac{n}{m} \in \mathcal{Q}$

$$(\frac{m}{n})^{+} = \max\{\frac{a}{b} \in \mathcal{Q} \mid \frac{a}{b} < \frac{m}{n} \text{ and } b < n\}$$

and

$$(\frac{m}{n})^{-} = \min\{\frac{a}{b} \in \mathcal{Q} \mid \frac{a}{b} > \frac{m}{n} \text{ and } b < n\}.$$

Just to avoid confusion we stress the fact that the minimum and the maximum of the above definition are taken w.r.t. the natural order of $\mathcal{Q}$. We are ready to formally define the family of embeddings ε_k.

Definition 2.4 Let $a = (a_{\frac{0}{1}}, a_{\frac{1}{1}}, ..., a_{\frac{k-1}{k}})$ be an element of $\mathcal{F}^{(k)}$, then we define:

$$\varepsilon_k(a) := (a_{\frac{0}{1}}, a_{\frac{1}{1}}, ..., a_{\frac{i}{k}}, a_{\frac{1}{k+1}}, ..., a_{\frac{k}{k+1}})$$

where for all $1 \leq j \leq k+1$ such that $\frac{j}{k+1} \in \mathcal{Q}$, we let $a_{\frac{j}{k+1}}$ be the solution of the linear equation:

$$\frac{\frac{j}{k+1} - (\frac{j}{k+1})^{-}}{(\frac{j}{k+1})^{+} - (\frac{j}{k+1})^{-}} = \frac{a_{\frac{j}{k+1}} - a_{(\frac{j}{k+1})^{-}}}{a_{(\frac{j}{k+1})^{+}} - a_{(\frac{j}{k+1})^{-}}}$$

Lemma 2.5 *ε_k is an embedding from $\mathcal{F}^{(k)}$ to $\mathcal{F}^{(k+1)}$.*

P r o o f. To see that the fractions $a_{\frac{j}{k+1}}$ of Definition 2.4 belong to S_{k+1}. Let $y = mx + n$ be the equation of the line connecting the points $((\frac{j}{k+1})^{+}, a_{(\frac{j}{k+1})^{+}})$ and $((\frac{j}{k+1})^{-}, a_{(\frac{j}{k+1})^{-}})$, hence

$$a_{\frac{j}{k+1}} = m\frac{j}{k+1} + n = \frac{mj + n(k+1)}{k+1}.$$

Since $a_{\frac{j}{k+1}}$ obviously belongs to $[0,1]$ this gives $a_{\frac{j}{k+1}} \in S_{k+1}$. Finally ε_k is trivially one-to-one. □

We build now a direct system in the category of sets as follows. For $i \leq j \in \omega_0$ let us denote by ε_{ij} the embedding $\varepsilon_{j-1} \circ ... \circ \varepsilon_i : \mathcal{F}^{(i)} \to \mathcal{F}^{(j)}$. With this notation we have $\varepsilon_i = \varepsilon_{i(i+1)}$ and ε_{ii} is the identity map of $\mathcal{F}^{(i)}$ for all $i \in \omega_0$. It is clear that $\varepsilon_{ij} \circ \varepsilon_{jk} = \varepsilon_{ik}$ for $i \leq j \leq k$.

Hence $\{(\mathcal{F}^{(i)}, \varepsilon_{ij}) \mid i, j \in \omega_0 \text{ and } i \leq j\}$ is a direct system, let D be its direct limit. D can be seen as the quotient of the disjoint union $\biguplus\{\mathcal{F}^{(k)} : k \in \omega_0\}$ over the equivalence relation E defined by

$$xEy \text{ if, and only if, } x \in \mathcal{F}^{(i)}, y \in \mathcal{F}^{(j)} \text{ for some } i \leq j \in \omega_0 \text{ and } \varepsilon_{ij}(x) = y.$$

Lemma 2.6 *For any element $a \in D$ there exists a unique $i \in \omega$ and a unique infinite sequences $(a^{(i)}, a^{(i+1)}, ...)$ such that for all $k \geq j \geq i$:*

(i) *$a^{(i)}$ has no inverse image with respect to ε_{i-1};*

(ii) *there is exactly one $a^{(j)} \in \mathcal{F}^{(j)}$;*

(iii) *$\varepsilon_{jk}(a^{(j)}) = a^{(k)}$;*

(iv) *the E-equivalence class of $a^{(j)}$ is a.*

Vice versa, given a sequence which satisfies the conditions (i)-(iii) above there exists a unique $a \in D$ for which the condition (iv) is satisfied.

Proof. Any $a \in D$ is an equivalence class of the disjoint union $\biguplus\{\mathcal{F}^{(k)} : k \in \omega_0\}$, let us indicate it by $\{a^{(i)}\}_{i \in I}$, since the ε_{ij} are embeddings, any two elements of the equivalence class must belong to different components of the direct limit, hence any $a^{(i)} \in \mathcal{F}^{(i)}$. Thus, the linear order on the direct limit $(\mathcal{F}^{(1)}, \mathcal{F}^{(2)}, ...)$, induces a linear order with minimum on the equivalence class a, say $(a^{(i)}, a^{(i+1)}, ...)$. Note that if there exists $b \in \mathcal{F}^{(i-1)}$ such that $\varepsilon^{i-1}(b) = a^i$ then also b would belong to the sequence $(a^{(i)}, a^{(i+1)}, ...)$, which is a contradiction. So also condition (i) is satisfied. Finally notes that $a^{(j)}$ and $a^{(k)}$ belong to the same equivalence class if, and only if, $\varepsilon_{jk}(a^{(j)}) = a^{(k)}$. For the other direction note that any sequence $(a^{(i)}, a^{(i+1)}, ...)$ which satisfies (i)-(iii) is an E-equivalence class. Hence taking $a = \{a^{(i)}, a^{(i+1)}, ...\}$ leads to the desired element. □

Definition 2.7 Given any element $a \in D$ we will call the sequence given by Lemma 2.6, the **snake** of a. Given a snake $(a^{(i)}, a^{(i+1)}, ...)$ any sub-sequence of the form $(a^{(j)}, a^{(j+1)}, ...)$ for $j \geq i$ will be called a **subsnake**.

Of course a subsnake is not a snake but still satisfies conditions (ii) and (iii) of Lemma 2.6. Note also that a sequence can be the subsnake of at most one snake, as they partition the set D.

Lemma 2.8 *Let $a \in D$ and $(a^{(i)}, a^{(i+1)}, ...)$ be its snake, there is a unique McNaughton function $f(x)$ such that, if $k \geq i$ then $a^{(k)} = (f(q))_{q \prec p}$ for some $p \in \mathcal{Q}$.*

Proof. Let $a^{(i)} = (a_{\frac{0}{1}}, a_{\frac{1}{1}}, ...)$ seen as the sequence of points in the plane $(\frac{0}{1}, a_{\frac{0}{1}}), \frac{1}{1}, a_{\frac{1}{1}})....$ Then take as f the function which linearly connects these points, this is obviuosly a McNaughton function. Since ε_k extends $a^{(k)}$ by interpolation we have that for all $k \geq i$, $a^{(k)} = (f_k(q))_{q \prec p}$, for a suitable $p \in \mathcal{Q}$. The unicity is straightforward as $(a^{(i)}, a^{(i+1)}, ...)$ is an infinite sequence. □

The final step is to endow D with the structure of MV-algebras. It easy to see that the embeddings preserve the constant 0 and $\neg$, hence we only need to introduce the operation $\oplus$ on D. Roughly speaking we will show that even if the ε_k are not MV-embeddings, they become MV-embeddings for a sufficiently large k. Such an idea is formalised in the following lemma.

Lemma 2.9 *Let $a, b \in D$ and let $(a^{(i)}, a^{(i+1)}, ...)$ and $(b^{(j)}, b^{(j+1)}, ...)$ their respective snakes. If $i \leq j$ then for some l the infinite subsequence $(a^{(j+l)} \oplus b^{(j+l)}, a^{(j+l+1)} \oplus b^{(j+l+1)}, ...)$ of $(a^{(j)} \oplus b^{(j)}, a^{(j+1)} \oplus b^{(j+1)}, ...)$ is a subsnake of a (necessarily unique) snake.*

Proof. Let $f(x)$ and $g(x)$ be the two McNaughton functions of $(a^{(j)}, a^{(j+1)}, ...)$ and $(b^{(j)}, b^{(j+1)}, ...)$, given by Lemma 2.8 and let us call $h(x)$ the McNaughton function $f(x) \oplus g(x)$. So

$$\left(a^{(j)} \oplus b^{(j)}, a^{(j+1)} \oplus b^{(j+1)}, ...\right) = \left((h(\frac{m}{n}))_{\frac{m}{n} \in Q^{\prec}, n \leq j}, (h(\frac{m}{n}))_{\frac{m}{n} \in Q^{\prec}, n \leq j+1}, ...\right)$$

Suppose that an l as in the statement does not exist. This amount to say that for all $k \in \omega$, $\varepsilon_{j+k}(a^{(j+k)} \oplus b^{(j+k)}) \neq a^{(j+k+1)} \oplus b^{(j+k+1)}$ and by the above equation this means that for $k \in \omega$,

$$\varepsilon_{jk}((h(\frac{m}{n}))_{\frac{m}{n} \in Q^{\prec}, n \leq j+k}) \neq (h(\frac{m}{n}))_{\frac{m}{n} \in Q^{\prec}, n \leq j+k+1}.$$

But, since ε_j is defined as an interpolation extension, this goes against the fact that $h(x)$ is built by finitely many linear pieces. □

Definition 2.10 We define the operation $\oplus$ in D as follows: let $a, b \in D$ and let $(a^{(i)}, a^{(i+1)}, a^{(i+2)}, ...)$ and $(b^{(j)}, b^{(j+1)}, b^{(j+2)}...)$ their respective snakes then $a \oplus b$ is defined as the element of D whose snake is inside $(a^{(j)} \oplus b^{(j)}, a^{(j+1)} \oplus b^{(j+1)}, ...)$.

Of course the above introduced operation on D we have used the classical symbols of MV-algebra, but so far we have not proved that $\langle D, \oplus, \neg, 0\rangle$ is an MV-algebra. We will do it now, proving directly that such a structures is isomorphic to the 1-generated free MV-algebra.

Theorem 2.11 *The algebra $\langle D, \oplus, \neg, 0\rangle$ is isomorphic to the MV-algebra $\langle M, \oplus, \neg, 0\rangle$ of all McNaughton function in one variable.*

Proof. We define an isomorphism Ψ form D to M, by taking as $\Psi(d)$ the McNaughton function associated to d as in Lemma 2.8. To prove that Ψ respects 0 and $\neg$ is straightforward.
In order to see that Ψ respects $\oplus$ take $c = (c^{(i)}, c^{(i+1)}...)$ and $d = (d^{(j)}, d^{(j+1)}...)$ in $\mathcal{D}$ with $j \geq i$. Suppose that $f = \Psi(c)$ and $g = \Psi(d)$, then it is readily seen that $f \oplus g$ is such that for all $k \geq j$, $a^{(k)} \oplus d^{(k)} = ((f \oplus g)(q))_{q \in \mathcal{Q}^{\prec}}$. But such McNaughton function is unique and it is exactly the image of $c \oplus d$. □

3 The general case

Summing up what we have done in the previous section, we have accomplished the following results:

- **:** In Definition 2.3 we have defined a family of embeddings going from the free algebra of $\mathbb{MV}_n$ and the free algebra of $\mathbb{MV}_{n+1}$.
- **:** In Lemma 2.6 we have shown that the equivalence classes of the direct limit have peculiar properties which allow to put them in bijective correspondence with McNaughton functions (Lemma 2.8).
- **:** Taking advantage of such a correspondence we defined MV-operations on the direct limit which make it isomorphic to the free MV-algebra (Theorem 2.11).

So the main issue is to define the embeddings in such a way that this bijective correspondence is guaranteed. This was solved in the 1-generator case by *attaching* a McNaughton function to a tuple, in such a way that this is preserved under the embeddings.

The difficulties which arise in the general case lay on the fact that while the dimension increases polynomialy in the number of generators (n+1), the number of adjacent points to *interpolate* grows exponentially (2^n).

To better explain the problem let us consider the free algebra with two generators of MV_1. In the Figure 3 we see that for the same tuple in $\mathcal{F}^{(1)}(2)$ there are at least two *natural* McNaughton functions which we can associate and, even worse, that none is linear but only pice-wise linear.

So in the cases with $m > 1$ there is not a straightforward *canonical* method to send a tuple of $\mathcal{F}^{(n)}(m)$ in a McNaughton function. For this reason in the rest of the paper our argument will take a more abstract flavour.

The first thing to do is to fix a precise (but still arbitrary) way to organize the element of a tuple in $\mathcal{F}^{(n)}(m)$ as points of the $(m+1)$-space.

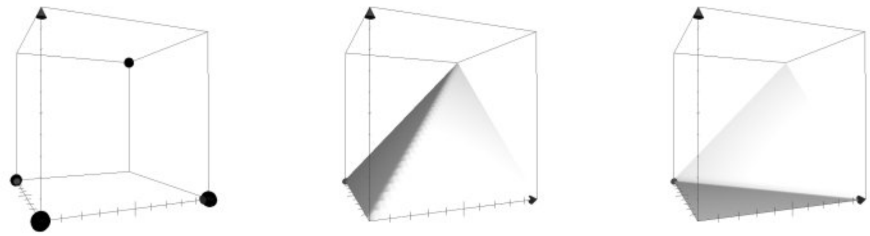

Fig. 3 Two possible McNaughton functions associated to a tuple in $\mathcal{F}_1(2)$

Definition 3.1 The linear order $\langle \mathcal{Q}^n, \prec^* \rangle$ is defined as follow:

- $\mathcal{Q}^n$ is the subset of direct product of n copies of $\mathcal{Q}$ given by those tuples for which the l.c.m of the denominators of its elements is the denominator of some of its elements (remember that we think of $0 = \frac{0}{1}$ and $1 = \frac{1}{1}$.)
- The order $\prec^*$ is inherited form $\prec$ as follows:

$$\begin{array}{ll} (x_1, ..., x_n) \prec^* (y_1, ..., y_n) \text{ if, and only if,} & \\ \qquad x_1, ..., x_n \prec y_1 \text{ or ... or } x_1, ..., x_n \prec y_n & \\ \text{Or, if this is not the case, then} & x_1 \prec y_1, \\ \qquad \text{or if } x_1 = y_1 \text{ then} & x_2 \prec y_2, \\ \qquad \vdots & \vdots \\ \qquad \text{or if } x_1 = y_1, ..., x_{n-1} = y_{n-1} \text{ then} & x_n \prec y_n. \end{array}$$

Just to give an example a $\prec^*$-listing of $\mathcal{Q}^2$ would be

$$(0,0),(0,1),(1,0),(1,1),(0,\frac{1}{2}),(1,\frac{1}{2}),(\frac{1}{2},0),(\frac{1}{2},1),(\frac{1}{2},\frac{1}{2}),(0,\frac{1}{3}),(1,\frac{1}{3}),(\frac{1}{3},0),$$
$$(\frac{1}{3},1),(\frac{1}{3},\frac{1}{3}),(0,\frac{2}{3}),(1,\frac{2}{3}),(\frac{1}{3},\frac{2}{3}),(\frac{2}{3},0),(\frac{2}{3},1),(\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{2}{3}),...$$

Note that the order of Definition 3.1 is arbitrarily chosen, any order which gives to elements of $\mathcal{S}_n$ "coordinates" with denominator dividing n would do the job.

Such an order allows to arrange an element of $\mathcal{F}^{(n)}(m)$ as a sequence of points in the $(m+1)$-space as follows. Since $\langle \mathcal{Q}^m, \prec^* \rangle$ is denumerable and a discrete linear order with an initial point it is order-isomorphic to $\langle \mathbb{N}, < \rangle$,

Definition 3.2 Let $\tau_m : \mathbb{N} \to \mathcal{Q}^m$ be the only bijection preserving the order and let σ_m be its inverse.

Then, given an element a_i of a tuple $(a_1, a_2, ..., a_k) \in \mathcal{F}^{(n)}(m)$, we can think of it as a point of the $(m+1)$-space $\tau_m(i).(a_i)$ where . denotes the operation of concatenation.

The next step will be the following: given a tuple $(a_1, a_2, ..., a_k) \in \mathcal{F}^{(n)}(m)$ we need to find a *canonical* McNaughton function passing through the sequence of points associated to it. The function must be canonical in the sense that if we prolong the sequence of points associated to the tuple with new points belonging to the range of the function and we re-apply our procedure to such a sequence we will obtain the same McNaughton function. This will give us a way of defining the embeddings $\varepsilon_{k,k+1}$ satisfying the requirement presented at the beginning of this section.

The idea is to associate McNaughton functions to elements of $\mathcal{F}^{(n)}(m)$ in a recursive way. This will allow to associate new functions only to the elements of $\mathcal{F}^{(k)}(m)$ which are not in the image of any previously used McNaughton function. So, to define $\varepsilon_{k,k+1}$ on sequences which are in the range of $\varepsilon_{k-1,k}$ we look beckwards to the McNaughton function associated to the tuple where the object comes from.

Definition 3.3 We say that a total function Φ which goes from $\bigcup_n \mathcal{F}^{(n)}(m)$ in the set of m-ary McNaughton functions is a **strict correspondence** if for any $(a_1, a_2, ..., a_k) \in \mathcal{F}^{(n)}(m)$ (with $k = \sum_{i=1}^{n} v_m(i)$) $\Phi(a_1, a_2, ..., a_k)$ is a McNaughton in m variables which has a minimal number of linear pieces among the ones whose graph contains $\{\tau(a_1).a_1, \tau(a_2).a_2, ..., \tau(a_k).a_k\}$.

The reason for requiring strictness is to be found in the constraints in order to consider *all* the McNaughton functions, as it will be clear form the proof of Lemma 3.8. Of course a strict correspondence in general does not satisfies the canonicity condition described above. But any strict correspondence induces another one which satisfies canonicity.

Definition 3.4 Let Φ be a strict correspondence and $(a_1, a_2, ..., a_k) \in \mathcal{F}^{(n)}(m)$ with $k = \sum_{i=1}^{n} v_m(i)$, we define $\mathbf{f}^{\Phi}_{(a_1,a_2,...,a_k)}$, the **McNaughton function associated to** $(a_1, a_2, ..., a_k)$ by induction on n as follows.

- If $n = 1$ then $\mathbf{f}^{\Phi}_{(a_1,a_2,...,a_k)} := \Phi(a_1, a_2, ..., a_k)$
- If $n > 1$ then let $h = \sum_{i=1}^{n-1} v_m(i)$, there are two cases:
  - (i) the points $\{\tau_m(a_{h+1}).(a_{h+1}), ..., \tau_m(a_k).(a_k)\}$ belong to the graph of $\mathbf{f}^{\Phi}_{(a_1,a_2,...,a_h)}$, then $\mathbf{f}^{\Phi}_{(a_1,a_2,...,a_k)} := \mathbf{f}^{\Phi}_{(a_1,a_2,...,a_h)}$.
  - (ii) otherwise $\mathbf{f}^{\Phi}_{(a_1,a_2,...,a_k)} := \Phi(a_1, a_2, ..., a_k)$.

It is straightforward to prove, by induction on n, that the map which sends a tuple in the McNaughton function associated to it is a strict correspondence.

Henceforth we assumed fixed a strict correspondence Φ and drop the superscript of $\mathbf{f}$.

Definition 3.5 Let $h = \sum_{i=1}^{n} v_m(i)$, $k = \sum_{i=1}^{n+1} v_m(i)$ and $a = (a_1, ..., a_h) \in \mathcal{F}^{(n)}(m)$ we define the tuple

$$\varepsilon^m_{n(n+1)}(a_1, ..., a_h) = (a_1, ..., a_h, a_{h+1}, ..., a_k).$$

The first h values are exactly as in a and if $i > h$ then a_i is such that $(\tau_m(i), a_i)$ belongs to to the graph of $\mathbf{f}_{(a_1,a_2,...,a_h)}$.
The embedding $\varepsilon^m_{nn'}$ with $n > n'$ is defined as the composition $\varepsilon^m_{n'(n'+1)} \circ ... \circ \varepsilon^m_{(n-1)(n)}$

Lemma 3.6 *The functions $\varepsilon^m_{nn'}$ are embeddings from the underlying sets of $\mathcal{F}^{(n)}(m)$ to $\mathcal{F}^{(n')}(m)$.*

That $\varepsilon^m_{nn'}$ are injective follows trivially from the definition. Note that in this case $\varepsilon^m_{nn'}$ are not $\neg$-embeddings because the arbitrariness of the choice of the strict correspondence destroys the symmetry needed for $\varepsilon^m_{n,n'}$ to be a $\neg$-embedding. The case with 1 generator does not suffer this problem as there exists only one correspondence satisfying strictness.

So, for every m, we have a directed system $\{(\mathcal{F}^{(i)}(m), \varepsilon^m_{ij}) \mid i, j \in \omega_0 \text{ and } i \leq j\}$. Let D_m the direct limit of such a system. Again D_m can be seen as the quotient of the disjoint union $\biguplus\{\mathcal{F}^{(k)}(m) \mid k \in \omega_0\}$, over the equivalence relation E defined by

$$xEy \text{ if, and only if, } x \in \mathcal{F}^{(i)}(m), y \in \mathcal{F}^{(j)}(m) \text{ for some } i \leq j \in \omega_0 \text{ and } \varepsilon_{ij}(x) = y.$$

Lemma 3.7 *For any element $a \in D_m$ there exists a unique $i \in \omega$ and a unique infinite sequences $(a^{(i)}, a^{(i+1)}, ...)$ such that for all $k \geq j \geq i$:*

(i) *$a^{(i)}$ has no inverse image with respect to ε_{i-1};*

(ii) *there is exactly one $a^{(j)} \in \mathcal{F}^{(j)}(m)$;*

(iii) *$\varepsilon_{jk}(a^{(j)}) = a^{(k)}$;*

(iv) *tthe E-equivalence class of $a^{(j)}$ is a.*

Vice versa, given a sequence which satisfies the conditions (i)-(iii) above there exists a unique $a \in D_m$ for which the condition (iv) is satisfied.

Proof. Exactly as in Lemma 2.6 □

So we can define, also in the general case, the notions of snake and subsnake.

Lemma 3.8 *For any snake $a = (a^{(i)}, a^{(i+1)}, ...)$ there exists a unique McNaughton function $f(x)$ such that if $k \geq i$ then there exists $p \in \langle \mathcal{Q}^n, \prec^* \rangle$ such that $a^{(k)} = (f(q))_{q \prec p}$. Vice versa, for any McNaughton function f there exists unique snake which a subsnake of the sequence $(f(q))_{q \in \mathcal{Q}^n}$.*

Proof. It is easy to see, from how the embedding ε are defined, that the McNaughton functions associated to $a^{(j)}$ for $i \leq j$ are all identical. The unicity comes from the fact that the sequence is infinite.

For the other direction it is sufficient to show that for any McNaughton function f there exists a a tuple in some $\mathcal{F}^{(n)}$ such that f is the unique McNaughton functions whose range contains the points associated to the tuple and its number of pieces is minimal. Take a tuple a such that f is minimal and suppose that is not unique. The McNaughton functions with minimal number of pieces passing through a are easily constructed by all the possible triangulation of the points associated to a, hence they are in a finite number. Now extend the tuple a with new points which rule out all the other minimal McNaughton functions. Finally take the appropriate b in some $\mathcal{F}^{(n)}$ which extends the sequence constructed. Then f is the McNaughton function associated to b. □

Lemma 3.9 *Let $a, b \in D_m$ and let $(a^{(i)}, a^{(i+1)}, ...)$ and $(b^{(j)}, b^{(j+1)}, ...)$ their respective snakes. If $i \leq j$ then for some l the infinite subsequence $(a^{(j+l)} \oplus b^{(j+l)}, a^{(j+l+1)} \oplus b^{(j+l+1)}, ...)$ of $(a^{(j)} \oplus b^{(j)}, a^{(j+1)} \oplus b^{(j+1)}, ...)$ is a subsnake. Similarly there exists m such that the infinite subsequence $(\neg a^{(j+m)}, \neg a^{(j+m+1)}, ...)$ of $(\neg a^{(j)}, \neg a^{(j+1)}, ...)$ is a subsnake*

Proof. As in the proof of Lemma 2.9, let f and g be the McNaughton functions given by Lemma 3.8 and consider the snake of $f \oplus g$. Similarly for $\neg$. □

Definition 3.10 We endow the set D_m with the structure of an MV-alegbra. Let $a, b \in D_m$ and let $(a^{(i)}, a^{(i+1)}, a^{(i+2)}, ...)$ and $(b^{(j)}, b^{(j+1)}, b^{(j+2)}, ...)$, respectively, their snakes.

- The element $\neg a$ is defined as the element of D_m whose snake is inside the sequence $(\neg a^{(i)}, \neg a^{(i+1)}, ...)$;
- the element $a \oplus b$ is defined as the element of D_m whose snake is inside the sequence $(a^{(j)} \oplus b^{(j)}, a^{(j+1)} \oplus b^{(j+1)}, ...)$;
- finally the constant 0 is interpreted in the snake composed by only sequences of 0's.

Theorem 3.11 *The algebra $\langle D_m, \oplus, \neg, 0\rangle$ is isomorphic to the MV-algebra $\langle M_m, \oplus, \neg, 0\rangle$ of all McNaughton function in m variable.*

Proof. As in the case with 1 generator (cfr. Theorem 2.11) □

4 Conclusion

We have described the free MV-algebras on m generators as a weak form of direct limit in which the operations of MV-algebras are restored in a geometrical way. This representation has the advantage of connecting the variety of all MV-algebras to the simpler varieties $\mathbb{MV}^{(n)}$. An immediate corollary of this construction is the following well-known result [5].

Corollary 4.1 *A sentence is valid in the Łukasiewicz infinite-valued logic if, and only if, for any $k \geq 1$ there is $n \geq k$ such that the sentence is valid in the Łukasiewicz n-valued logic.*

P r o o f. For the non-trivial direction, let us see that if a sentence is not valid in Łukasiewicz infinite-valued logic then there exists k such that it is not valid in all Łukasiewicz n-valued logic with $n \geq k$. Since the Lindemaum algebra is isomorphic to the algebra of McNaughton functions, the failure of a formula in Łukasiewicz calculus can be characterised by the fact that its associated McNaughton function is not the constantly 1 function. This in turn implies that the snake associated to it by Theorem 3.11 is not constantly one, hence there exists a free algebra $\mathcal{F}^{(k)}(m)$ such that the equivalence class associated to the sentence is not 1, and by the definition of the embeddings ε this also holds for all for all the free algebras $\mathcal{F}^{(n)}(m)$ with $n \geq k$. □

Another fact worth to be mentioned is that even if we did not give a direct proof that $\langle D_m, \oplus, \neg, 0 \rangle$ is an MV-algebra, it follows from Theorem 3.11 that indeed it is. This means that if instead of proving theorem Theorem 3.11 we undertook the tedious task of verifying that all MV-axioms hold in D_m it would have turned out to be an MV-algebra and easy proved to be the free MV-algebra over m generators (using the Theorem above). But then the proof of Theorem 3.11 would have given us an alternative proof of McNaughton Theorem.

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