A(nother) duality for the whole variety of MV-algebras

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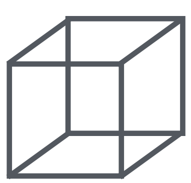
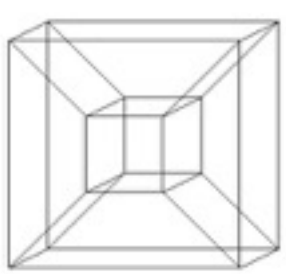
Preliminary notes

- It is a work in progress.
- MV-algebras as case study, but the main result applies in more general cases.



An n-cube



An infinite dimensional cube

Finitely presented MV-algebras

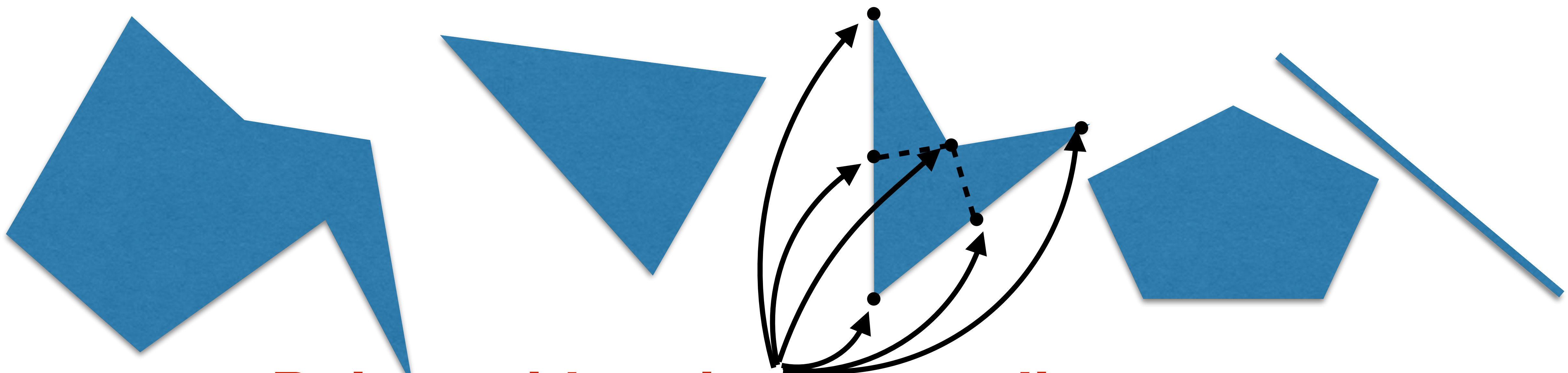
A finitely presented algebra is the quotient of a finitely generated free algebra over a finitely generated congruence

$$\frac{\mathcal{F}(n)}{\langle\{s_1 = t_1, s_2 = t_2, ..., s_m = t_m\}\rangle}$$

The equations $s_1=t_1,...,s_m=t_m$ define a closed subspace of $[0,1]^n$

Finitely presented MV-algebras

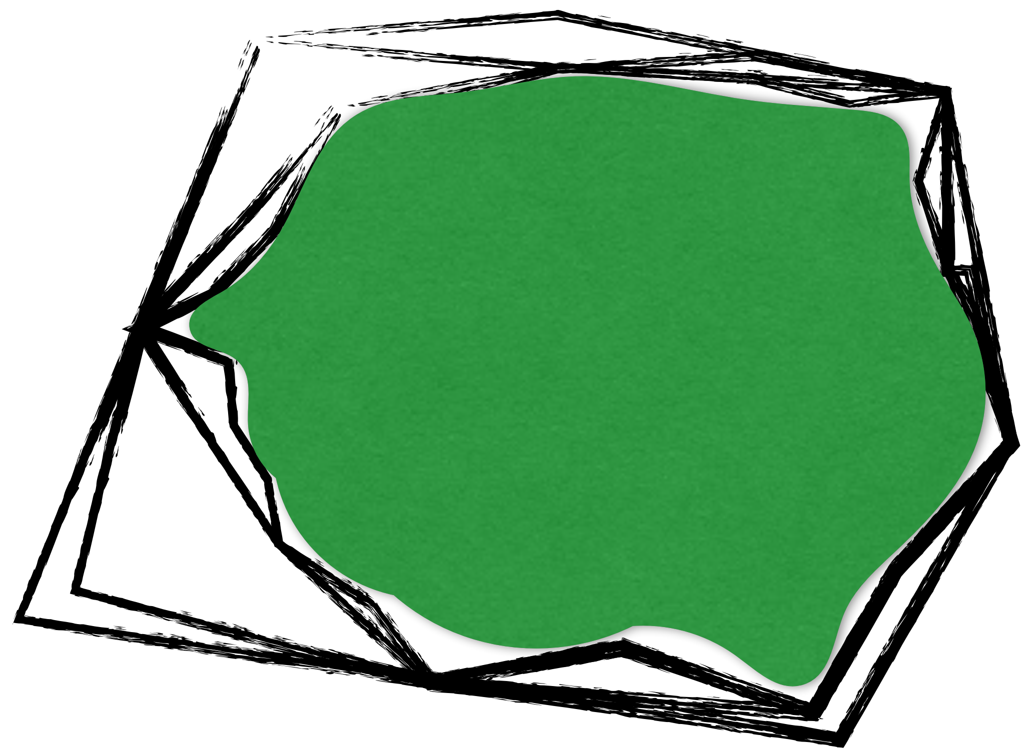
In the case of MV-algebras, those equations define a **rational polyhedron.**



Points with rational coordinates

More precisely, a rational polyhedron is a finite union of convex hulls of rational points in $[0,1]^n$.

Semisimple MV-algebras



Closed spaces in $[0,1]^n$ are *limits* of rational polyhedra

Z-maps

Let a, b possibly infinite cardinals. A continuous map

$$z = (z_d)_{d<b} : [0,1]^a \longrightarrow [0,1]^b$$

is called a Z-map if for each d<b, z_d is **piecewise linear with integer coefficients**.

In other words, if there is a **finite number** of (affine) linear polynomials with integer coefficients

$$l_1, \dots, l_{i(d)}$$

such that for every point x in $[0,1]^a$ there is j <i(d) with $z_i(x) = l_j(x)$.

Given subsets P in $[0,1]^a$ and Q in $[0,1]^b$, a Z-map z : P —> Q is a restriction of Z-map from $[0,1]^a$ into $[0,1]^b$

The duality for semisimple and finitely presented MV-algebras

Theorem

The category of semisimple MV-algebras with their homomorphisms

is dually equivalent

to the category of closed subspaces of $[0,1]^{a}$, with *a* ranging among all cardinals, and Z-maps as arrows.

In particular, the category of **finitely presented MV-algebras** with their homomorphisms

is dually equivalent

to the category P_Z of rational polyhedra and Z-maps.

MV-algebras (general case)

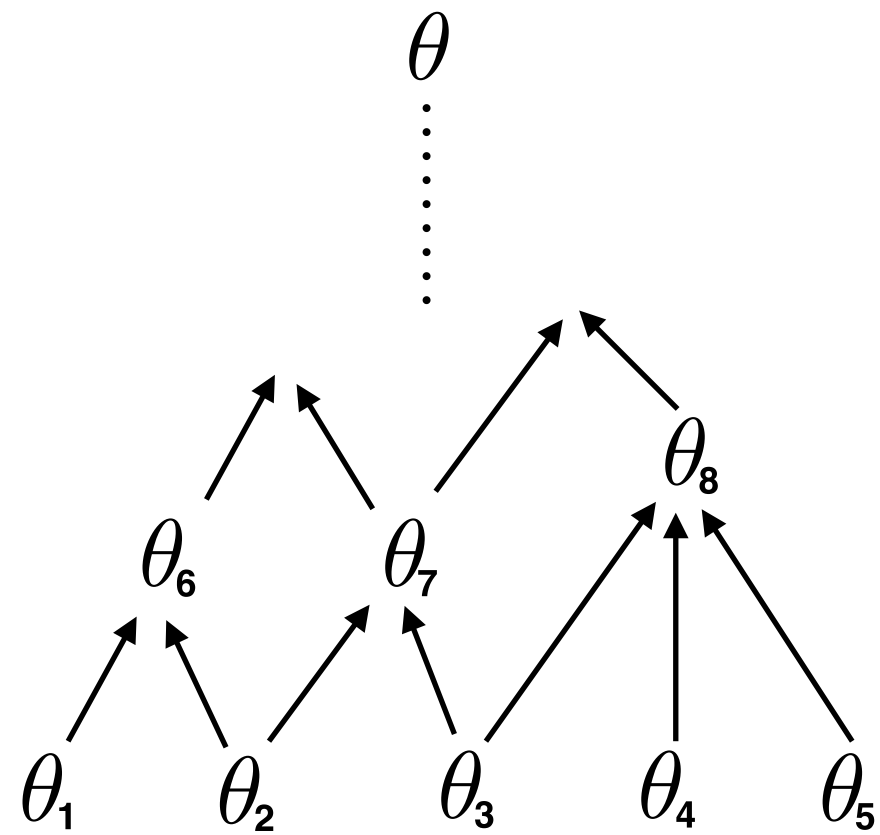
Any algebra is the quotient of a free algebra over some congruence

$$\frac{\mathcal{F}(\kappa)}{\langle \{s_i = t_i\}_{i \in I} \rangle}$$

Finitely presented algebras as building blocks

Start with any algebra $\dfrac{\mathcal{F}(\kappa)}{\theta}$

One can form a directed diagram by taking all **finite subsets** of θ

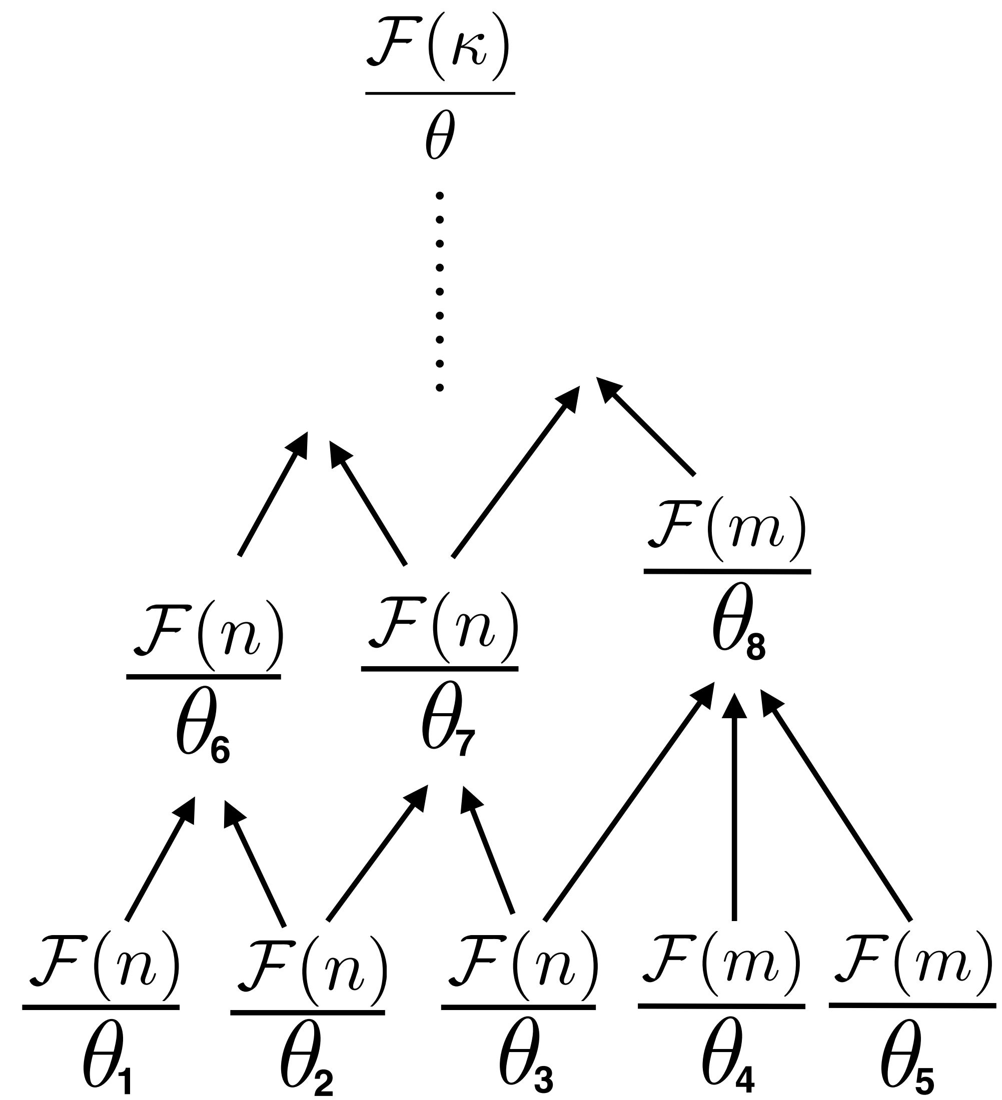


Finitely presented algebras as building blocks

Start with any algebra $\frac{\mathcal{F}(\kappa)}{\theta}$

One can form a directed diagram by taking all finite subsets of θ

This corresponds to a directed diagram of algebras



Directedness of the diagram

$$\frac{\mathcal{F}(X_1, X_2)}{\langle s_1(X_1) = t_1(X_1), s_2(X_2) = t_2(X_2) \rangle}$$

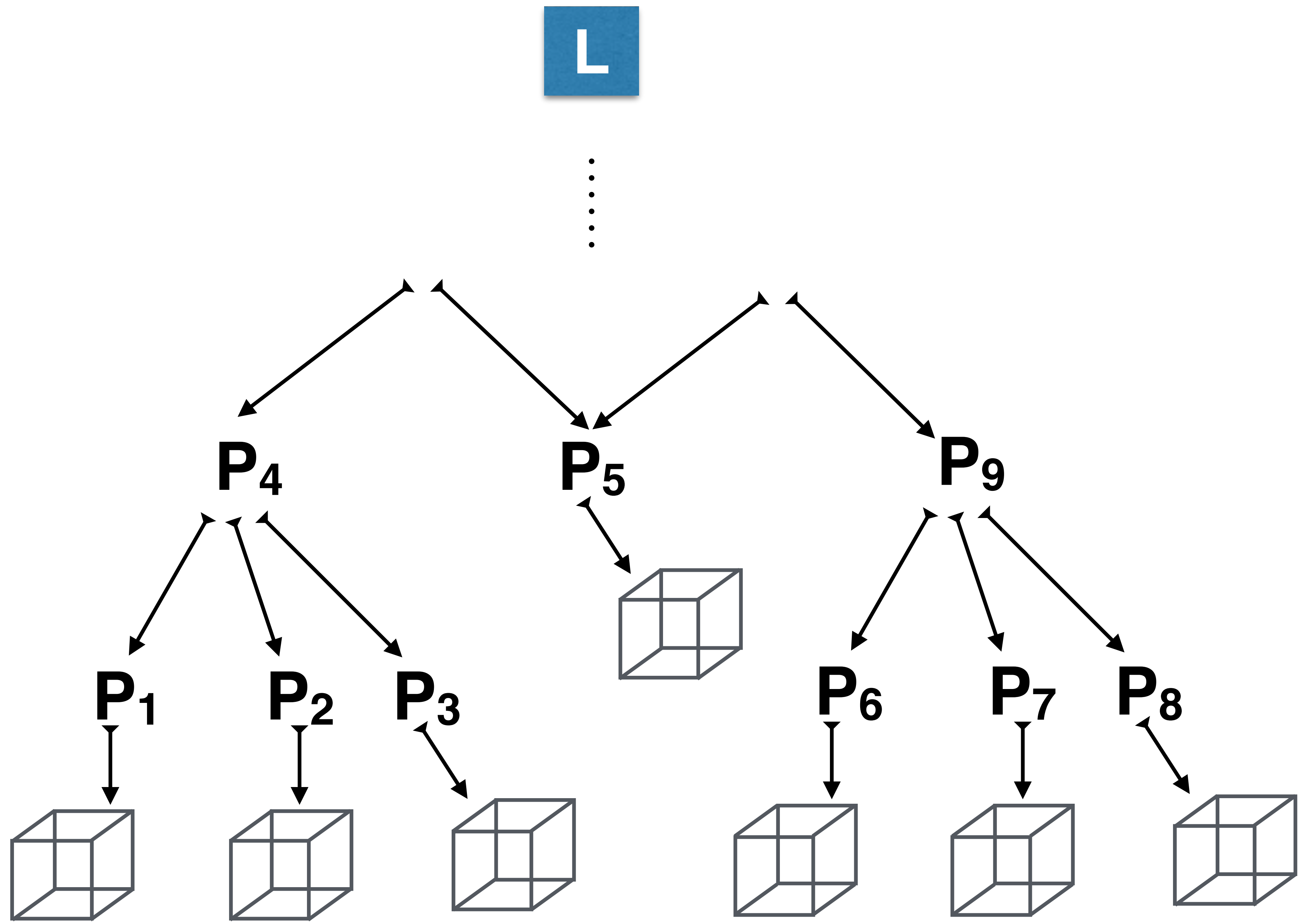
$$\nearrow \qquad\qquad \nwarrow$$

$$\frac{\mathcal{F}(X_1)}{\langle s_1(X_1) = t_1(X_1) \rangle} \qquad\qquad \frac{\mathcal{F}(X_2)}{\langle s_2(X_2) = t_2(X_2) \rangle}$$

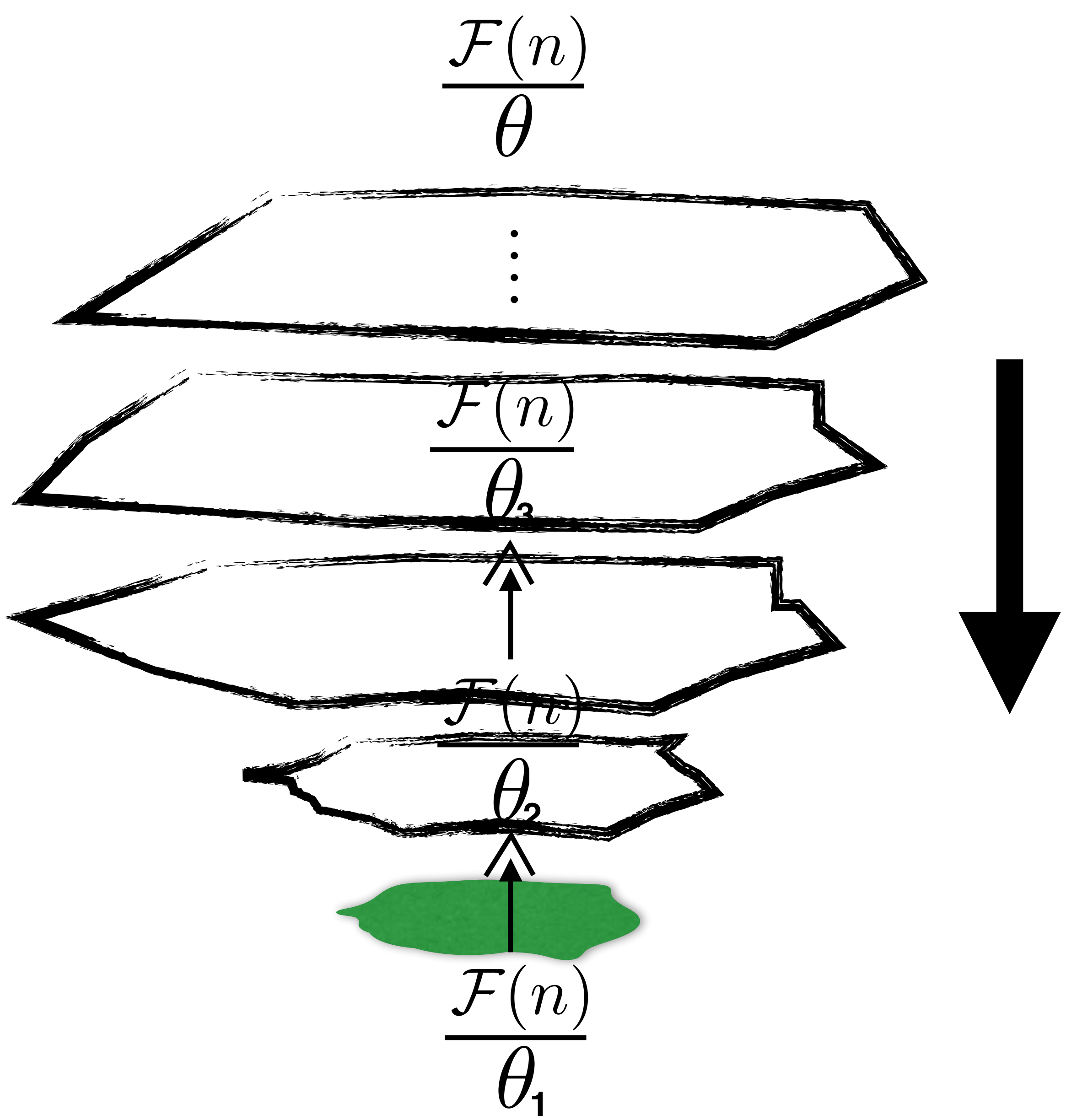
$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$\mathcal{F}(X_1) \qquad\qquad\qquad\qquad \mathcal{F}(X_2)$$

Limits of rational polyhedra



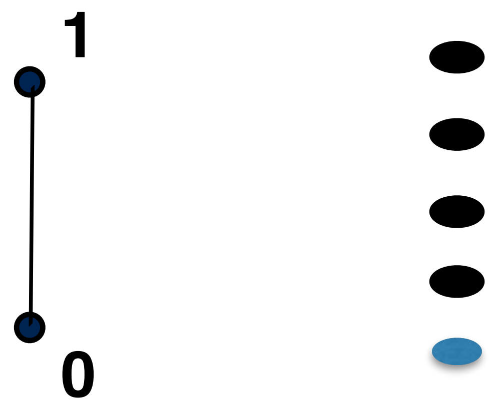
Finitely generated MV-algebras



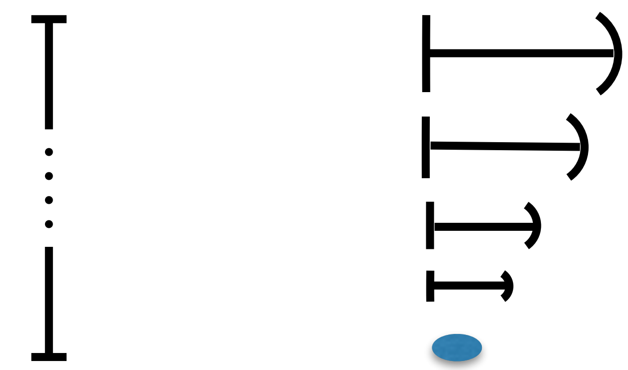
For finitely generated MV-algebra, it is enough to consider diagrams that have the order type of ω

Four examples

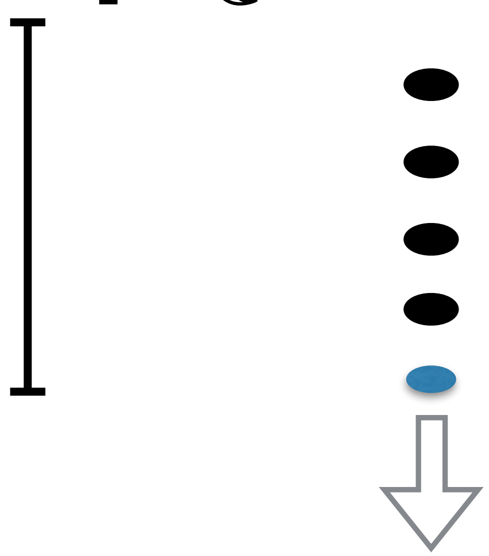
The algebra {0,1}



Chang's algebra

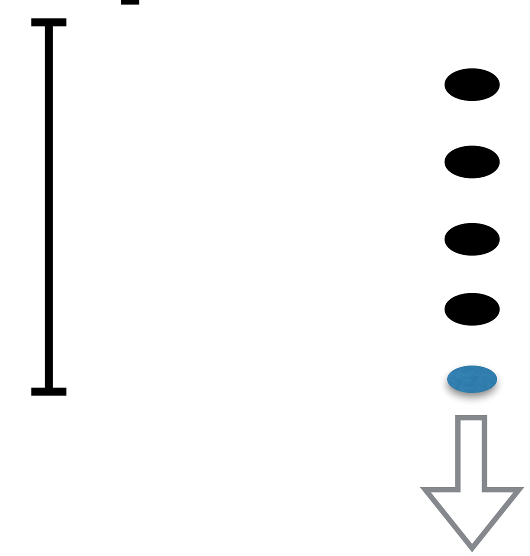


$[0,1] \cap \mathbb{Q}$



$[0,1]^{\omega}$

$[0,1]$



$[0,1]^{2^{\omega}}$

Ind- and pro- completions

- The **ind-completion** of a category C is a new category whose objects are **directed diagrams in C**.

- Arrows in **ind-C** are **family of equivalence classes** of arrows in C.

- The **pro-completion** is formed similarly.

Ind- and pro- completions

Let B and C be two categories

if $B \simeq C$ then ind-$B \simeq$ (pro-$C^{op})^{op}$.

Now, $MV_{fp} \simeq (P_Z)^{op}$, so

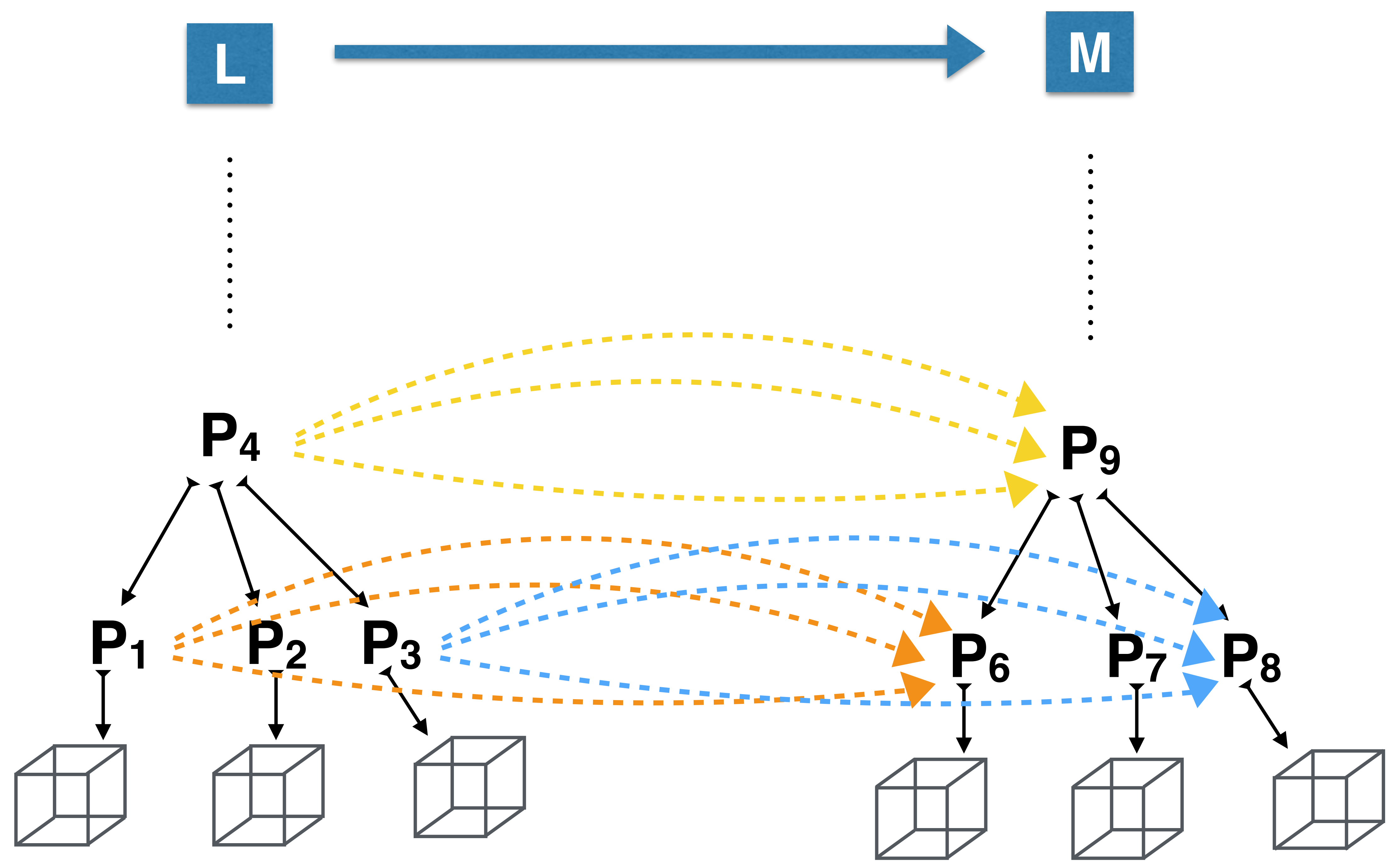
MV $\simeq$ ind-$MV_{fp} \simeq$ ((pro-$(P_Z)^{op})^{op})^{op} \simeq$ **(pro-$P_Z)^{op}$**.

The dual of MV

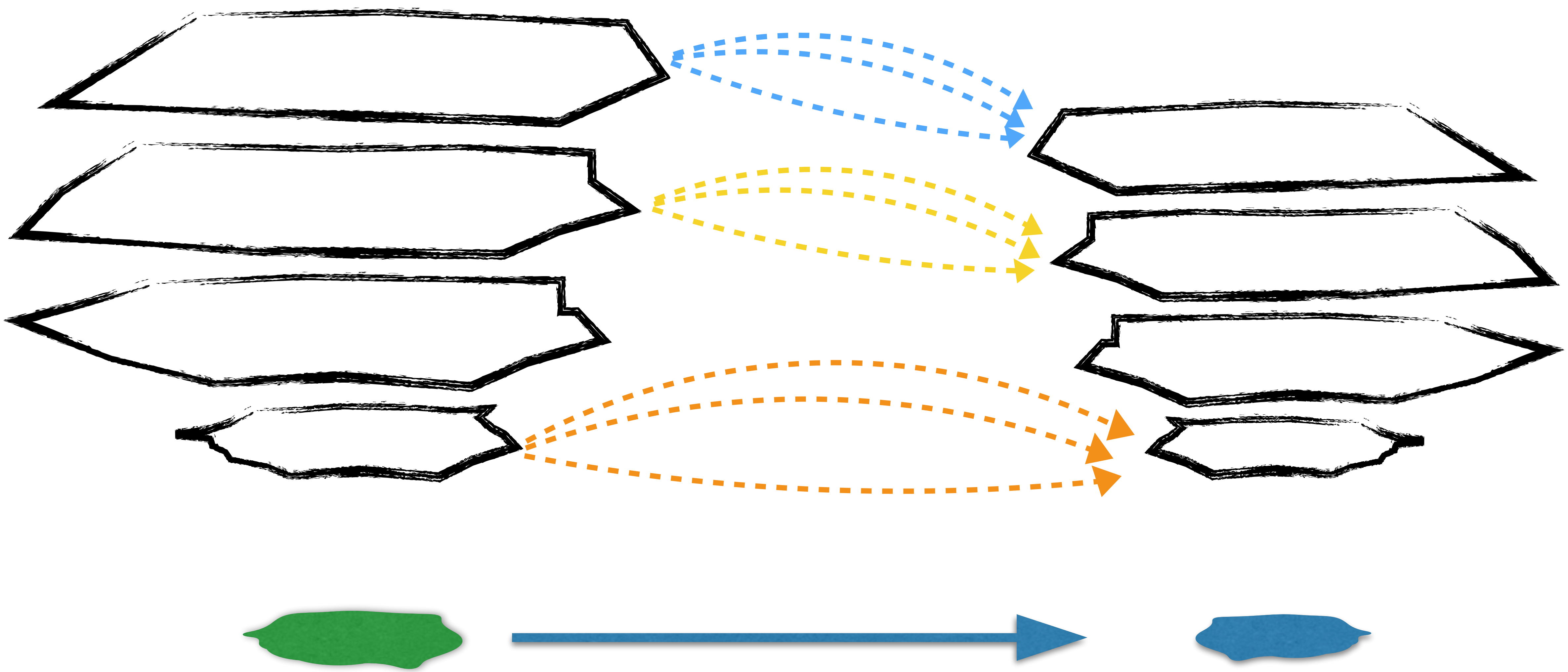
Theorem

$$\mathrm{MV} \simeq (\text{pro-}\mathrm{P}_{\mathrm{Z}})^{\mathrm{op}}$$

Arrows in the pro-completion



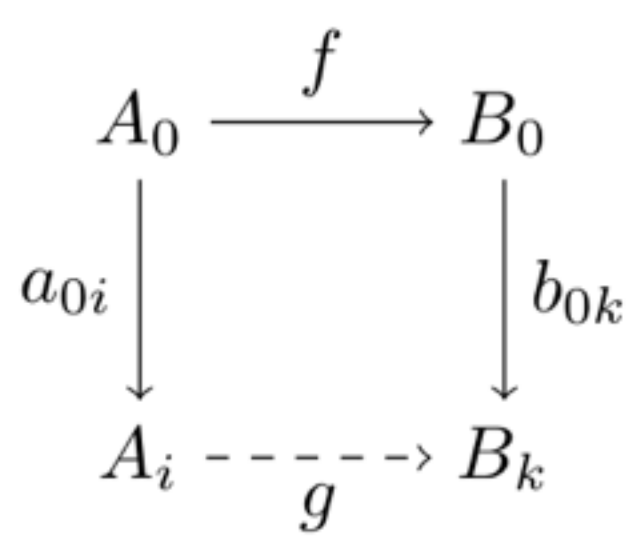
Arrows in the finitely generated case



Compatible arrows

Let $A = \{(A_i, a_{ij}) \mid i,j \in \omega\}$ and $\{(B_k, b_{kl}) \mid k,l \in \omega\}$ be a pair of diagrams of finitely presented algebras. We can assume that A_0 is $[0,1]^n$ and B_0 is $[0,1]^m$.

The family of **compatible arrows** $C(A,B)$ is given by all arrows $f : A_0 \longrightarrow B_0$ for which for any i there exists k such that $b_{0k} \circ f$ factors through A_i.

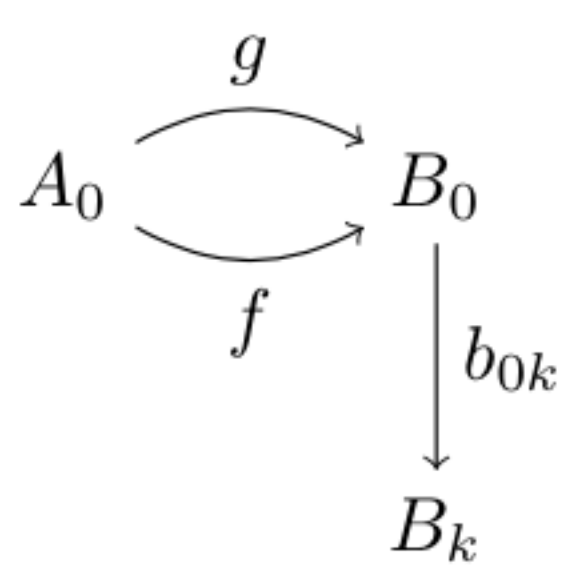


Eventually equal maps

Let C be a category and A = $\{(A_i,a_{ij}) \mid i,j \in \omega\}$ and $\{(B_k,b_{kl}) \mid k,l \in \omega\}$ be a pair of diagrams of finitely presented algebras.

We define an equivalence relation E on C(A,B) as follows.

Two arrows $f,g \in$ C(A,B) are in E (to be read as f and g being **eventually equal**), if, and only if, there exists $k \in K$ such that $b_{0k} \circ f = b_{0k} \circ g$.

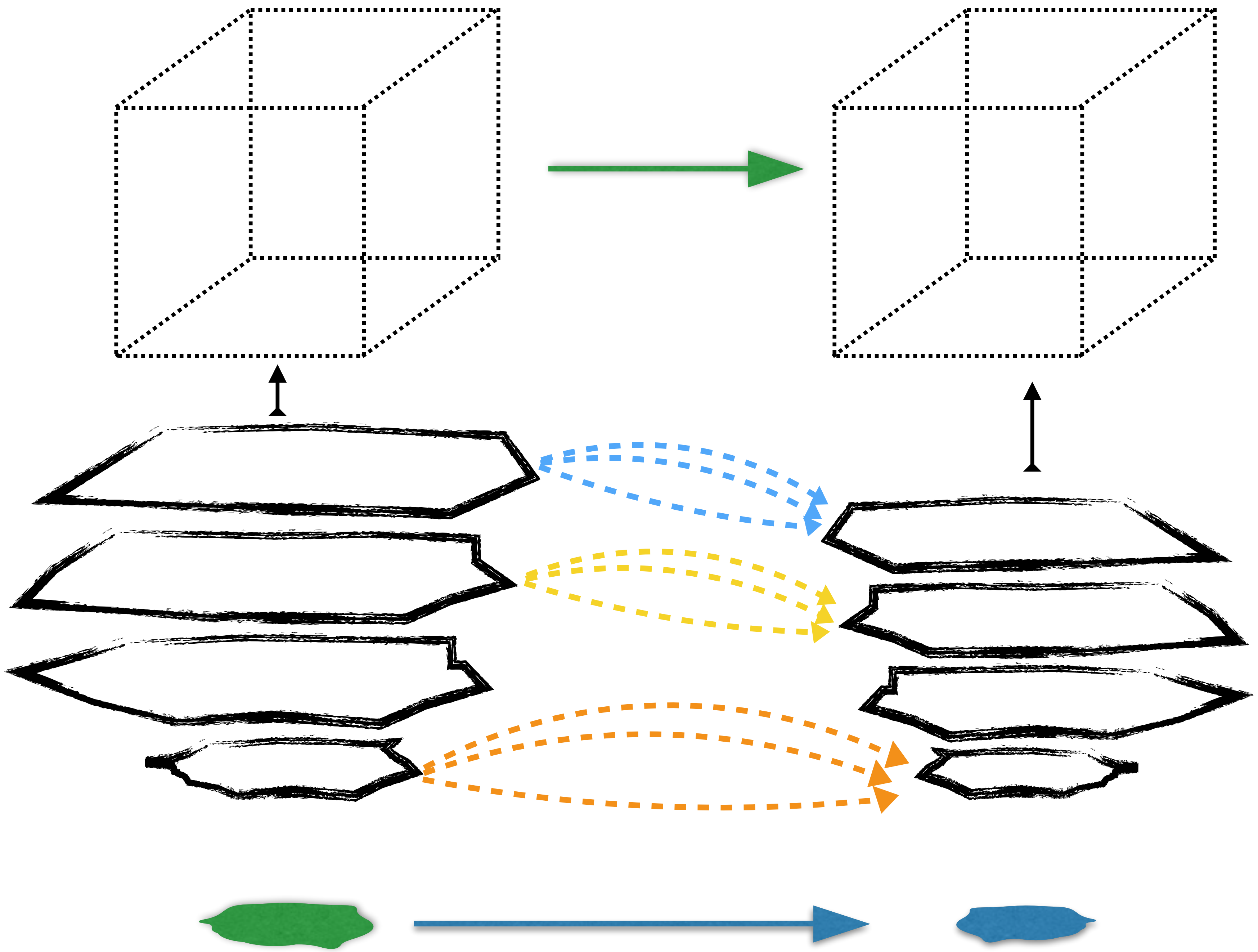


The case of finitely generated algebras

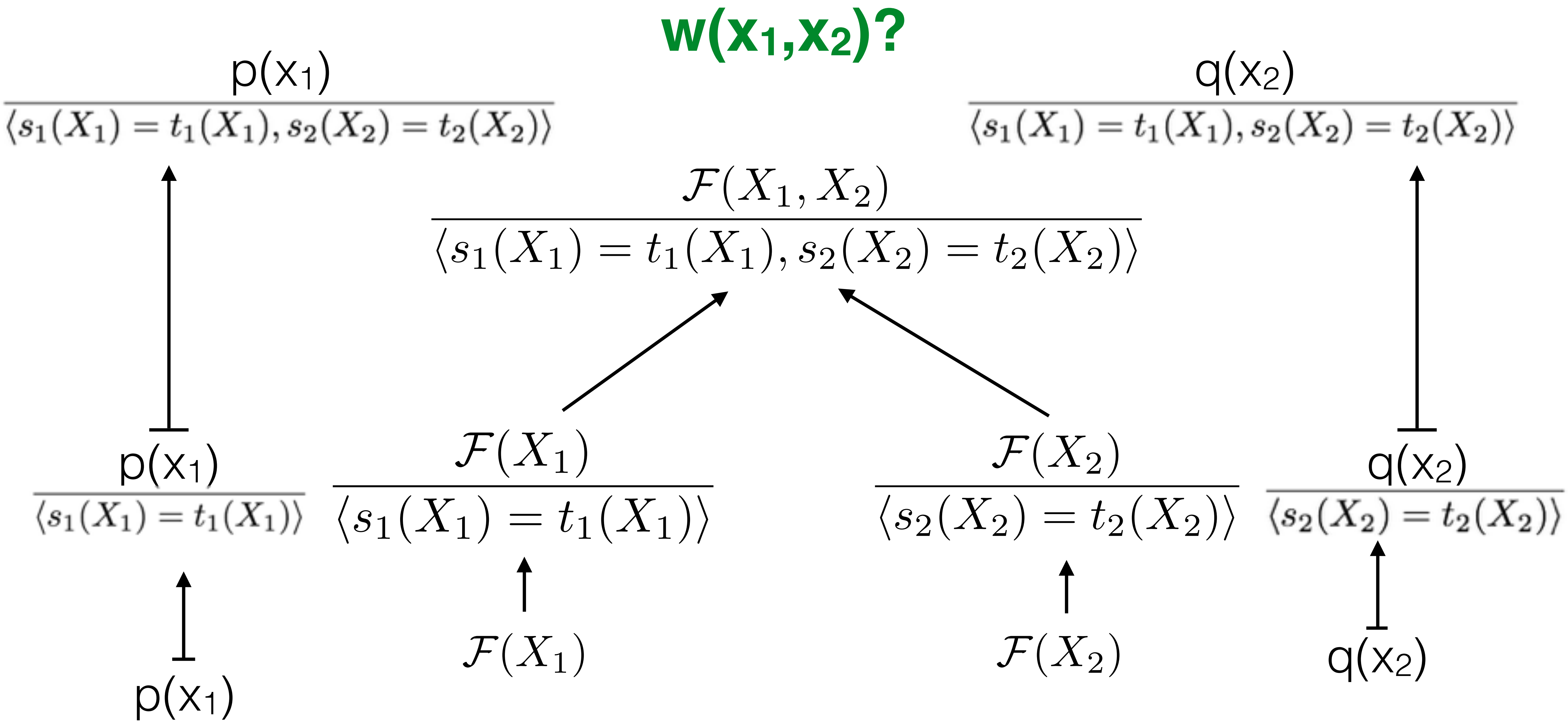
Theorem *Let $\{(A_i, a_{ij}) \mid i, j \in I\}$ and $\{(B_{kl}, b_{kl}) \mid k, l \in K\}$ be diagrams of order type ω in a category* C, *A and B their respective limits in* ind-C, *and suppose that the arrows a_{ij} and b_{kl} are epic.*

1. *For any $\mathcal{E}$-equivalence class C in $\mathcal{C}(A, B)$ of arrows $f \colon A_0 \to B_0$ there is a corresponding arrow ϕ_C between A and B in* ind-C.
2. *Vice-versa, for any arrow $\phi = \{\phi_i\}_{i \in I}$ in* ind-C *between A and B, there is an $\mathcal{E}$-equivalence class C_ϕ of arrows $f \colon A_0 \to B_0$ in $\mathcal{C}(A, B)$.*
3. *The above associations are such that $C = C_{\phi_C}$ and $\phi = \phi_{C_\phi}$.*

Arrows in the finitely generated case

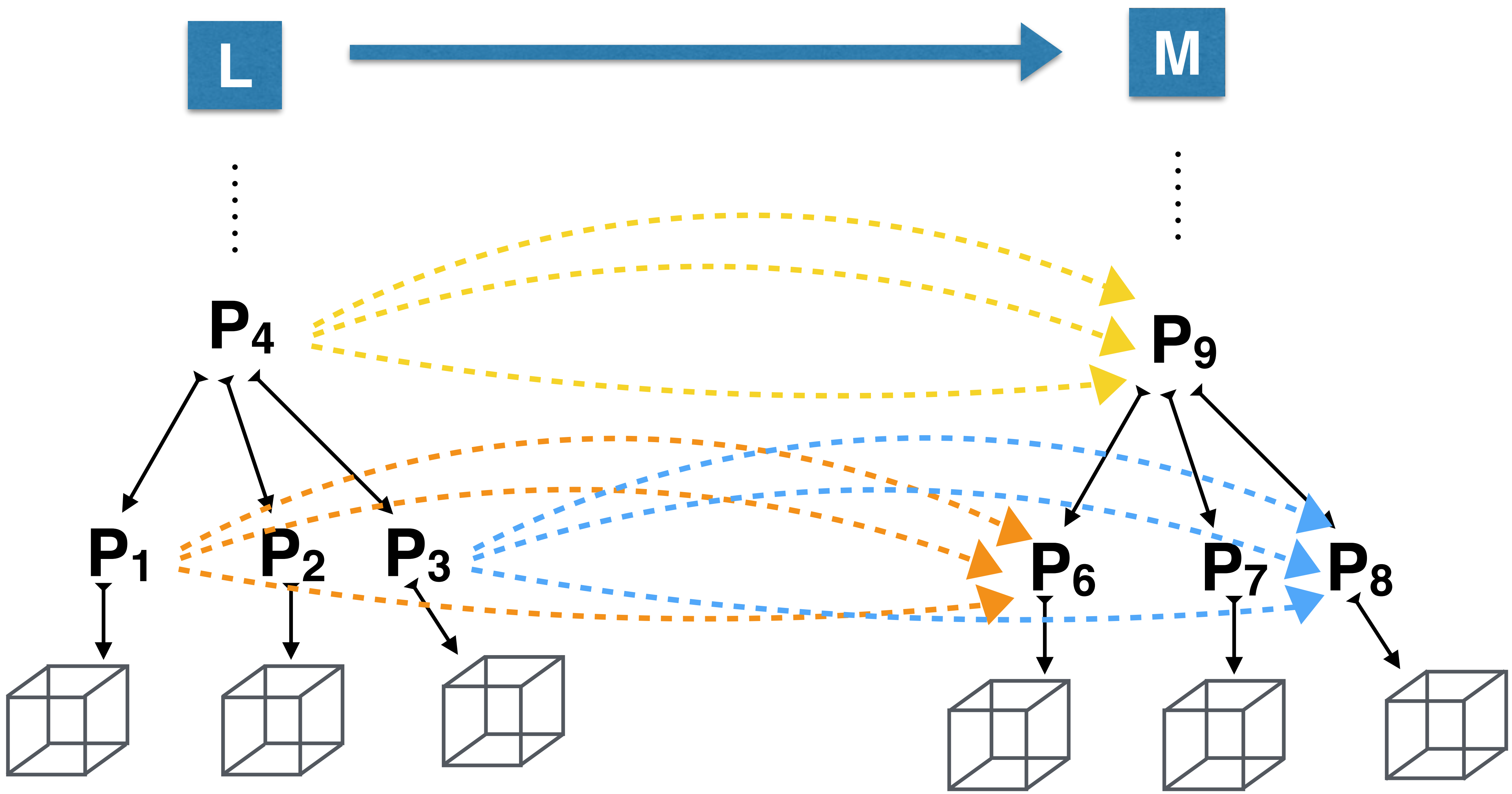


How epic are arrows in the general diagram?

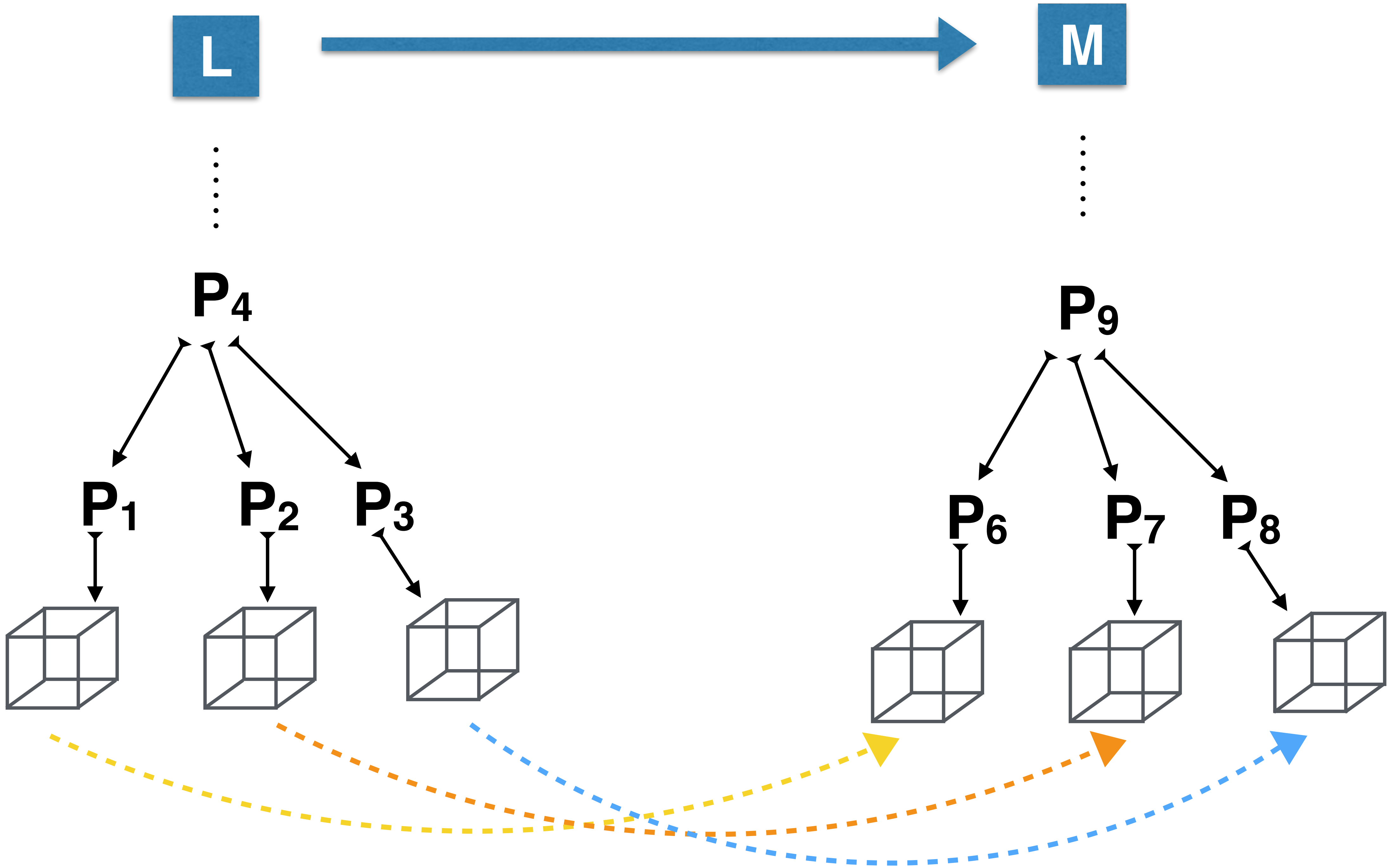


Arrows are *jointly* epic

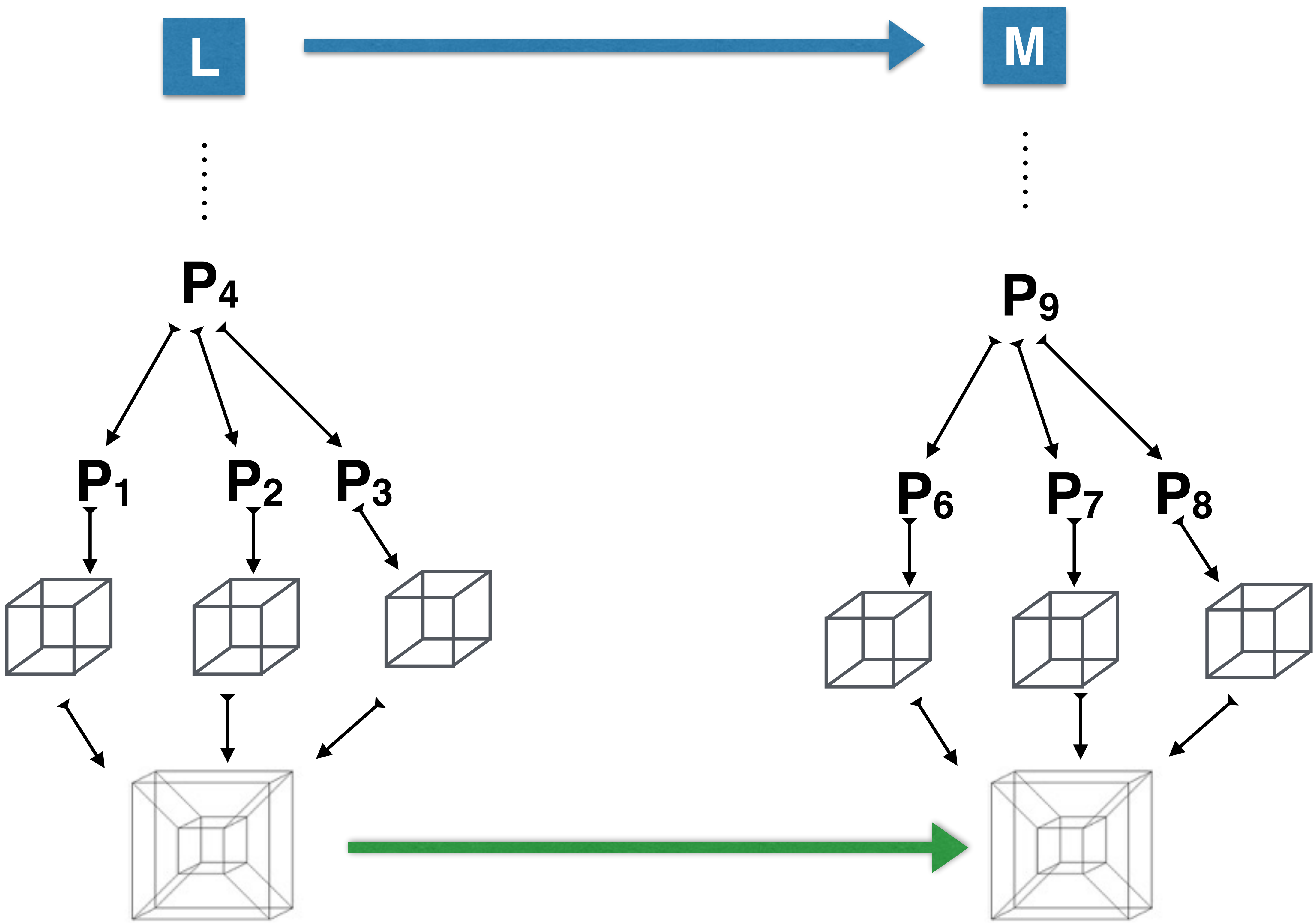
Arrows in the pro-completion



Arrows in the pro-completion



Arrows in the pro-completion



Thank you for your attention!