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Forcing in Łukasiewicz Predicate Logic

Abstract. In this paper we study the notion of forcing for Łukasiewicz predicate logic (Ł∀, for short), along the lines of Robinson's forcing in classical model theory. We deal with both finite and infinite forcing. As regard to the former we prove a Generic Model Theorem for Ł∀, while for the latter, we study the generic and existentially complete standard models of Ł∀.

Keywords: First order Many-valued logic, Łukasiewicz Logic, Robinson Forcing

1. Introduction

Łukasiewicz and Tarski introduced in [21] a generalization of classical propositional logic with infinite truth values. This system, called Łukasiewicz logic, was independently shown to be complete by Rose and Rosser [28], and by Chang [5, 6]. While Rose and Rosser's proof was basically syntactic, Chang showed that Łukasiewcz logic is complete w.r.t. the variety of MV-algebras. In his proof Chang shows that the variety of MV-algebras is generated by the MV-algebra whose lattice reduct is the real unit interval $[0, 1]$ and the basic operations are defined as

$$x \oplus y = \min\{x + y, 1\} \quad \text{and} \quad x^* = 1 - x.$$

For this reason Łukasiewicz logic can be also seen as a member of the family of fuzzy logics based on *triangular norms* [12] (i.e. binary, associative, commutative, monotone operations over $[0, 1]$ having 1 as a neutral element). Indeed, the operation $\odot$, defined as $x \odot y = (x^* \oplus y^*)^*$ corresponds to the t-norm $x \odot y = \max(x + y - 1, 0)$, while $x \Rightarrow y = x^* \oplus y = \min(1 - x + y, 1)$ is the residual implication of $\odot$. The structure $\langle [0, 1], \odot, \Rightarrow, 0, 1\rangle$ is often referred as the *standard MV-algebra*.

So the study of Łukasiewicz logic has different and relevant interests. On the one hand the study of its algebraic semantics has led to many important results as well as links with other well established fields of mathematics. Just to cite a few of them we recall Mundici's categorical equivalence between MV-algebras and lattice ordered groups with strong unit [23], with its plethora of consequences; and McNaughton Theorem [22], which gives a description of the free MV-algebras as algebras of continuous piece-wise linear functions, offering a geometrical interpretation of Łukasiewicz logic.

On the other hand Łukasiewicz logic plays a central role in the formal treatment of uncertainty and, in particular, in Hájek's programmatic approach to a rigorous formulation of foundations for fuzzy logic. In this work we try to give our contribute to the latter development, introducing tools for the investigation of first order Łukasiewicz logic.

While propositional Łukasiewicz logic has been deeply studied from the logical, algebraic, categorical, functional and computational point of view (see [9], for an account on many of these topics), that is not the case for Łukasiewicz predicate logic. Indeed the predicate version of Łukasiewicz logic is not tame as the propositional fragment, as witnessed by the number of negative results already present in the literature.

Łukasiewicz predicate logic is not complete w.r.t. standard models and, as shown in [25] (see also [12]), it is undecidable and its set of standard tautologies is Π_2-complete and hence it is not recursively axiomatizable [29]. In [2] it is proved that the Lindenbaum algebra of Łukasiewicz predicate logic is not semi-simple (for a refined analysis of completeness results in Łukasiewicz logic refer to [3]).

Nevertheless Łukasiewicz predicate logic has raised interest since its introduction, allowing, for instance, a coherent axiomatization of set theory with full comprehension [7, 11]. The study of a set theory based on many-valued logics has nowadays a renewed importance as a possible foundational approach to fuzzy logic. This is witnessed by many studies on set theory based on first order Łukasiewicz logic [13, 16] and the more general first order Basic logic [14, 15].

The notions we are going to introduce come from a generalization of techniques used in model theory, which, in turn, generalize Cohen's methods in set theory. For this reason we expect our results to be of some interest both to persons which are interested in the investigation of the semantics of Ł∀ and to people which work on set theory based on many-valued logic.

The reason why we have chosen the system Ł∀ has to be found in its involutive negation. As we will see further on, the forcing interpretation of the negation it is a delicate matter. Of course the concepts, and hopefully the results, contained in this paper can be further generalized to first order Basic logic, however in our proofs the involutive property of the negation plays often an important, hence such a generalization seems to be a challenging task.

A previous knowledge of classical forcing is not required to understand this article as we will build our definitions from scratch and fully detail the proofs. However the reader familiar with either the forcing in set theory or, better, the forcing in model theory will have a deeper comprehension of the motivations leading to such definitions and results.

In 1970 A. Robinson introduced finite and infinite forcing for classical predicate logic (see [1, 26, 27]). Finite forcing is a syntactical notion inspired by Cohen's forcing in set theory (the terminology derives from the fact that the conditions are finite sets of basic sentences).

A more general form of the finite forcing for infinitary logic was developed by H. J. Keisler in [18]. One of the main tool offered by forcing in model theory is the *Generic Model Theorem* which is a powerful method to construct models verifying a given set of formulas. Applying the Generic Model Theorem one can obtain new proofs of important results in classical model theory such as: the completeness theorem, the omitting types theorem, the interpolation theorems.

Infinite forcing is a semantical notion defined as a relation between structures and sentences; it provides classes of models that generalize the algebraic closed fields. For this reason it is a very useful tool to handle notions as model-completion, model-companion and existentially closed structures.

In view of the important achievements given by the classical forcing we would have had pleasure to give more applications of our results as well as sharper characterizations; however, apart for the pioneering article [10], we faced in many cases a surprising lack of basic notions in the model theory of Ł∀, which did not allowed us to even formulate problems or conjectures. For this reason we hope that the positive results which we have found exploring the notion of forcing in Łukasiewicz predicate logic will motivate a more systematic study of its semantics also using methods of classical model theory.

For reader ease we recall some notions and basic results on the classical finite and infinite forcing.

Let $\mathcal{L}$ be a first order language (without equality), with the connectives $\vee, \neg, \exists$ as primitive logical symbols. Consider a countable set C of new constants and denote by $\mathcal{L}(C)$ the language obtained from $\mathcal{L}$ by adding the constants of C.

Definition 1.1 ([18]). A forcing property is a triple $(P, \leq, f)$, where $(P, \leq)$ is a poset with a first element 0 and f is a function from P to the set of sets of atomic sentences of $\mathcal{L}(C)$ such that for all $p \leq q$ in P we have $f(p) \subseteq f(q)$.

The elements of P are called *conditions*.

Definition 1.2 ([18]). The forcing relation $p \Vdash \phi$ (p *forces* ϕ) between conditions in P and sentences of $\mathcal{L}(C)$ is defined by induction:

- If ϕ is atomic then $p \Vdash \phi$ iff $\phi \in f(p)$;

- $p \Vdash \neg\phi$ iff there is no $q \geq p$ such that $q \Vdash \phi$;
- $p \Vdash \phi \vee \psi$ iff $p \Vdash \phi$ or $p \Vdash \psi$;
- $p \Vdash \exists x \phi(x)$ iff $p \Vdash \phi(c)$, for some $c \in C$.

Definition 1.3 ([18]). *A* subset G of P is called generic if $q \leq p$ and $p \in G$ implies $q \in G$;

- For any $p, q \in G$ there is r in G such that $p \leq r$ and $q \leq r$;
- For any sentence ϕ in $\mathcal{L}(C)$ there is $p \in P$ such that $p \Vdash \phi$ or $p \Vdash \neg\phi$.

A structure $\mathfrak{A}$ is generated by a generic set G if every sentence ϕ of $\mathcal{L}(C)$ which is forced by some $p \in G$ holds in $\mathfrak{A}$. $\mathfrak{A}$ is generic for $p \in P$ if it is generated by a generic set G that contains p. $\mathfrak{A}$ is generic if it is generic for 0.

Theorem 1.4 (Generic Model Theorem, [18]). *Assume that the language $\mathcal{L}$ is countable and C is countable. If $(P, \leq, f)$ is a forcing property and $p \in P$ then there exists a generic structure for p.*

Now we shall present some basic material on Robinson infinite forcing. Let $\mathcal{L}$ be an arbitrary first order language and Σ an inductive class of $\mathcal{L}$-structures. For a structure $\mathfrak{A}$, $\mathcal{L}(\mathfrak{A})$ will denote the language obtained from $\mathcal{L}$ by adding the elements of $\mathfrak{A}$ as new constants.

Definition 1.5 ([27]). The infinite forcing relation $\mathfrak{A} \Vdash \phi$ between structures and sentences ϕ of $\mathcal{L}(\mathfrak{A})$ is inductively as follows:

- If ϕ is atomic then $\mathfrak{A} \Vdash \phi$ iff $\mathfrak{A} \models \phi$;
- $\mathfrak{A} \Vdash \neg\phi$ iff there is no extension $\mathfrak{B}$ of $\mathfrak{A}$ in Σ such that $\mathfrak{B} \Vdash \phi$;
- $\mathfrak{A} \Vdash \phi \vee \psi$ iff $\mathfrak{A} \Vdash \phi$ or $\mathfrak{A} \Vdash \psi$;
- $\mathfrak{A} \Vdash \exists x \phi(x)$ iff $\mathfrak{A} \Vdash \phi(a)$ for some $a \in \mathfrak{A}$.

A structure $\mathfrak{A}$ in Σ is called infinitely generic (in Σ) if for any sentence ϕ of $L(\mathfrak{A})$ we have $\mathfrak{A} \Vdash \phi$ or $\mathfrak{A} \Vdash \neg\phi$.

Theorem 1.6 ([27]). *Every structure $\mathfrak{A}$ on Σ can be embedded in a infinitely generic structure $\mathfrak{B}$ in Σ.*

For a succint survey on the results on forcing proved by Robinson and Barwise see [19].

This paper is structured as follows. In the next section we provide some basic notions and results on the syntax and the semantics of $Ł_\infty$ and $Ł\forall$.

Section 3 deals with the finite forcing for Ł∀, a syntactical notion that translates Keisler's forcing [18] to our framework. After proving some properties regarding finite forcing we define the finitely generic structures and prove a many-valued version Generic Model Theorem (**Theorem 3.8**).

In Section 4 we define the infinite forcing value of sentences in Łukasiewicz predicate logic. This is a semantical concept arising from Robinson's infinite forcing in classical model theory [27]. A number of results of this section describe the behavior of this new semantics w.r.t. the logical operations of Ł∀.

In Section 5 we introduce the infinitely generic structures and prove that any Ł∀ structure can be embedded in a generic one (**Corollary 5.8**). This provides the existence of generic structures, as well as the existence of existentially complete structures. Finally, we obtain a global characterization of the class of generic models (**Theorem 5.10**).

Section 6 contains a number of remarks and future lines of research.

We conclude our work with a brief appendix which describes some supplementary properties of infinite forcing.

2. Propositional and Predicate Łukasiewicz logic

In this section we recall some syntactical notions of Łukasiewicz logic (language, axiomatization, the deduction theorem, etc), as well as some elements of its semantics (MV-algebras, truth value, Ł∀ structures and models, theories, etc). The basic references are the books [9, 12].

Łukasiewicz propositional logic, $Ł_\infty$ for short, is defined from a countable set Var of propositional variables $p_1, p_2, \ldots, p_n, \ldots$, and two connectives $\rightarrow$ and $\neg$. The axioms of $Ł_\infty$ are the following:

$$\varphi \rightarrow (\psi \rightarrow \varphi); \qquad (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi));$$
$$((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi); \qquad (\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi),$$

where φ, ψ and χ are formulas. Modus ponens is the only rule of inference. The notions of proof and theorem are defined as usual.

The equivalent algebraic semantics for $Ł_\infty$ (in the sense of [4]) is given by the variety of MV-algebras, i.e. structures $\mathcal{A} = \langle A, \oplus, ^*, 0\rangle$ satisfying the following equations (see [9]):

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \qquad x \oplus y = y \oplus x, \qquad x \oplus 0 = x,$$
$$(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x, \qquad x^{**} = x, \qquad x \oplus 0^* = 0^*.$$

Other operations are definable as follows:

$$x \Rightarrow y = x^* \oplus y; \qquad x \odot y = (x^* \oplus y^*)^*;$$
$$\varphi \vee \psi = (\varphi \Rightarrow \psi) \Rightarrow \psi; \qquad \varphi \wedge \psi = (\varphi^* \vee \psi^*)^*;$$
$$\varphi \leftrightarrow \psi = (\varphi \Rightarrow \psi) \odot (\psi \Rightarrow \varphi);$$
$$\varphi^n = \underbrace{\varphi \odot ... \odot \varphi}_{n \text{ times}}; \qquad (n)\varphi = \underbrace{\varphi \oplus ... \oplus \varphi}_{n \text{ times}}.$$

A Ł∀ language $\mathcal{L}$ consists of the following primitive symbols:

- an infinite set V of variable symbols: $w, x, y, z, ...$;
- an arbitrary set of constant symbols;
- an arbitrary set of predicate symbols: with each predicate symbol P it is associated a natural number $ar(P) \geq 1$ (the arity of P);
- the connectives $\rightarrow, \neg$;
- the existential quantifier $\exists$;
- the parentheses: $(,), [,]$.

Often we shall say variable (resp. constant, predicate) instead of variable symbol (resp. constant, symbol, predicate symbol).

A *term* of $\mathcal{L}$ is a variable or a constant. An *atomic formula* has the form $P(t_1, ..., t_n)$ where P is an n-ary predicate and $t_1, ..., t_n$ are terms. The *formulas* of $\mathcal{L}$ are defined by induction:

- the atomic formulas are formulas;
- if φ and ψ are formulas then $\neg\varphi$ and $\varphi \rightarrow \psi$ are formulas;
- if φ is a formula and x is a variable then $\exists x\varphi$ is a formula.

We shall denote by **Form** the set of formulas of $\mathcal{L}$. We shall use the following abbreviations:

$$\varphi \underline{\odot} \psi = \neg(\varphi \rightarrow \neg\psi); \qquad \varphi \underline{\oplus} \psi = \neg\varphi \rightarrow \psi;$$
$$\varphi \underline{\vee} \psi = (\varphi \rightarrow \psi) \rightarrow \psi; \qquad \varphi \underline{\wedge} \psi = \neg(\neg\varphi \underline{\vee} \neg\psi);$$
$$\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \underline{\odot} (\psi \rightarrow \varphi); \qquad \forall x\varphi = \neg\exists x\neg\varphi;$$
$$\varphi^{\underline{n}} = \underbrace{\varphi \underline{\odot} ... \underline{\odot} \varphi}_{n \text{ times}}; \qquad \underline{(n)}\varphi = \underbrace{\varphi \underline{\oplus} ... \underline{\oplus} \varphi}_{n \text{ times}}.$$

An occurrence of a variable x in a formula φ is *free* if x does not belong to a subformula of φ having the form $\exists x\psi$. Otherwise, an occurrence of x in φ is *bound*. x is *free* in φ if some occurrence of x is free in φ. A *sentence* is a formula without free variables. We shall write $\varphi(x_1, ..., x_n)$ if all the free variables of φ are among $\{x_1, ..., x_n\}$. $FV(\varphi)$ will denote the set of free variables of φ. If φ is a formula, x is a variable and t is a term then by $\varphi(t)$ we mean the formula obtained by replacing all free occurrences of x in φ by t. A variable y is *substitutable* for x in φ if no subformula of φ having the form $\exists y\psi$ contains a free occurrence of x in φ. A term t is substitutable for x in φ if every variable of t is substitutable in φ.

The following are the axioms of Ł∀:

(A_0) the axioms of ∞-valued propositional Łukasiewicz calculus L_∞;

(A_1) $\forall x\varphi \to \varphi(t)$, where the term t is substitutable for x in φ;

(A_2) $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, where x is not free in φ;

Ł∀ has two *rules of inference*:

- Modus ponens: from φ and $\varphi \to \psi$, derive ψ;
- Generalization: from φ, derive $\forall x\varphi$.

The notions of formal proof, formal theorems, etc., are defined as usual.

If φ is a formal theorem of Ł∀ then we write $\vdash \varphi$. It is obvious that the formal theorems of L_∞ remain formal theorems of Ł∀. A *theory* is a set of formulas. If T is a theory and φ is a formula then the formal inference $T \vdash \varphi$ is defined in the usual way.

Notice that for Ł∀, the following form of the Deduction Theorem holds.

PROPOSITION 2.1 (Deduction Theorem). *If T is a theory of Ł∀, φ a sentence and ψ a formula then the following equivalence holds:*

$$T \cup \{\varphi\} \vdash \psi \text{ iff } T \vdash \varphi^{\underline{n}} \to \psi, \text{ for some integer } n \geq 1.$$

We review, now, the basic semantical notions for Ł∀. Let $\mathcal{L}$ be a Ł∀ language and $\mathcal{M}$ an MV-algebra. An $\mathcal{M}$-structure has the form $\mathfrak{A} = \langle A, (P^{\mathfrak{A}})_P, (c^{\mathfrak{A}})_C\rangle$ where

A is a non-empty set (the universe of the structure);

for any n-ary predicate P of $\mathcal{L}$, $P^{\mathfrak{A}} : A^n \to \mathcal{M}$ is an n-ary $\mathcal{M}$-valued relation on A;

for any constant c of $\mathcal{L}$, $c^{\mathfrak{A}}$ is an element of A.

Let $\mathfrak{A}$ be an $\mathcal{M}$-structure. An *evaluation* of $\mathcal{L}$ in $\mathfrak{A}$ is a function $e : V \to A$. For two evaluations e, e' of $\mathcal{L}$ in $\mathfrak{A}$ and for $x \in V$ we denote

$$e \equiv_x e' \text{ iff } e\mid_{V\setminus\{x\}} = e'\mid_{V\setminus\{x\}} .$$

For any term t and for any evaluation of $\mathcal{L}$ in $\mathfrak{A}$ we define the element $t^{\mathfrak{A}}(e)$ of A:

if t is the variable x then $t^{\mathfrak{A}}(e) = e(x)$;
if t is the constant c then $t^{\mathfrak{A}}(e) = c^{\mathfrak{A}}$.

For any evaluation $e : V \to A$ and for any formula φ of $\mathcal{L}$ we define by induction the element $\|\varphi(e)\|_{\mathfrak{A}}$ of $\mathcal{M}$ (also indicated by $\|\varphi(e)\|$ when there is no danger of confusion), called the *truth value* of φ:

- if φ is of the form $P(t_1, ..., t_n)$ then $\|\varphi(e)\| = P^{\mathfrak{A}}(t_1^{\mathfrak{A}}(e), ..., t_n^{\mathfrak{A}}(e))$;
- if $\varphi = \neg\psi$ then $\|\varphi(e)\| = \|\psi(e)\|^*$;
- if $\varphi = \psi \to \chi$ then $\|\varphi(e)\| = \|\psi(e)\| \Rightarrow \|\chi(e)\|$;
- if $\varphi = \exists x\psi$ then $\|\varphi(e)\| = \bigvee\{\|\psi(e')\| \mid e' \equiv_x e\}$.

We shall say that $\mathfrak{A}$ is a *safe* $\mathcal{M}$-structure if for any evaluation $e : V \to A$ and for any formula ψ of Ł∀, the supremum $\bigvee\{\|\psi(e')\| \mid e' \equiv_x e\}$ exists in $\mathcal{M}$ (in this case the infimum $\bigwedge\{\|\psi(e')\| \mid e' \equiv_x e\}$ also exists).

If φ is a sentence then $\|\psi(e)\|$ does not depend on e: in this case we denote $\|\varphi\| = \|\varphi(e)\|$.

Let $\varphi(x_1, ..., x_n)$ be a formula of $\mathcal{L}$. Then $\forall x_1 ... \forall x_n \varphi(x_1, ..., x_n)$ is a sentence. We define

$$\|\varphi(x_1, ..., x_n)\|_{\mathfrak{A}} = \|\forall x_1 ... \forall x_n \varphi(x_1, ..., x_n)\|_{\mathfrak{A}} = \bigwedge_{a_1,...,a_n \in A} \|\varphi(a_1, ..., a_n)\|_{\mathfrak{A}} .$$

A formula φ is a $\mathcal{M}$-*tautology* if $\|\varphi\|_{\mathfrak{A}} = 1$ for all safe $\mathcal{M}$-structures $\mathfrak{A}$. A safe $\mathcal{M}$-structure $\mathfrak{A}$ is a *model* of a theory T if $\|\varphi\|_{\mathfrak{A}} = 1$ for all $\varphi \in T$.

A *standard structure* is a $[0, 1]$- structure. A standard structure is always safe. A *standard model* of a theory T is a $[0, 1]$-structure which is a model of T.

PROPOSITION 2.2. *The formal theorems of Ł∀ are tautologies.*

PROOF. The usual induction on the length of a formal theorem. ■

THEOREM 2.3. *Any consistent theory T of Ł∀ has a standard model.*

3. Finite Forcing

In this section we introduce the finite forcing value of a formula, we study the finite generic structures and prove our Generic Model Theorem.

In order to compare the forcing value of a formula with its truth value we will deal here only with standard structures, so henceforth we will call structure any standard structure of Ł∀. Nevertheless one might want to consider a more general notion of forcing whose value ranges in an arbitrary complete MV-algebra, in this case our results will still hold, the choice of confining ourselves to standard structures has here only the purpose of a smooth presentation.

In classical model theory, finite forcing is expressed by a binary relation $p \Vdash \varphi$ between the elements p of a poset P (conditions) and the sentences of predicate logic. Traditionally, the conditions are particular finite sets of sentences. In order to extend the finite forcing to Ł∀, we shall replace the forcing relation $\Vdash$ by a binary $[0,1]$-valued relation between the elements of P and the sentences of Ł∀.

We fix a Ł∀ countable language $\mathcal{L}$ and we let C be a countable set of new constants; we will denote by $\mathcal{L}(C)$ the language extended with the new constants. Let E be set of sentences of $\mathcal{L}(C)$ and At its subset of atomic sentences. Let $(P, \leq)$ be a poset, α a cardinal and $\{a_\xi \ : \ \xi < \alpha\} \subseteq P$ such that for all ordinals $\xi < \eta < \alpha$ we have $a_\xi \leq a_\eta$. A set of the form $\{a_\xi \ : \ \xi < \alpha\}$ having this property will be called an *ordinal-indexed subset of* P.

Definition 3.1. A *forcing property* is a structure of the form $\langle P, \leq, 0, f\rangle$ such that the following properties hold:

(i) $(P, \leq, 0)$ is a poset with a first element 0;

(ii) Any ordinal-indexed subset of P has an upper bound;

(iii) $f : P \times At \to [0,1]$ is a function such that for all $p, q \in P$ and $\varphi \in At$ we have $p \leq q \Longrightarrow f(p,\varphi) \leq f(q,\varphi)$.

The elements of P are called *conditions*.

This definition generalizes the one given by Keisler in [18]. Notice that the requirement in (**i**) of the existence of a minimum element 0 is not necessary for our purposes in this article; nevertheless the concept of forcing property arises as a generalization of an arbitrary subset of the set of finite theories of $\mathcal{L}(C)$, hence the explicit presence of a symbol interpreting the empty set may turn to be usefull. This is easily seen if one thinks to classical forcing where the set of formulas *weakly forced* by the empty set plays a special role. Condition (**ii**) is a technical requirement which is essential to deal with forcing properties which are more than countable.

DEFINITION 3.2. Let $\langle P, \leq, 0, f\rangle$ be a forcing property. For any $p \in P$ and $\varphi \in E$ we define the real number $[\varphi]_p \in [0,1]$ by induction on the complexity of φ:

(1) if $\varphi \in At$ then $[\varphi]_p = f(p, \varphi)$;

(2) if $\varphi = \neg\psi$ then
$$[\varphi]_p = \bigwedge_{p \leq q} [\psi]^*_q;$$

(3) if $\varphi = \psi \to \chi$ then
$$[\varphi]_p = \bigwedge_{p \leq q} ([\psi]_q \Rightarrow [\chi]_p);$$

(4) if $\varphi = \exists x \psi(x)$ then
$$[\varphi]_p = \bigvee_{c \in C} [\psi(c)]_p.$$

The real number $[\varphi]_p$ is called the *finite forcing value* of φ at p.

The above definition is a generalization of Robinson forcing to the setting of many-valued logic. Notice that the part that most differs form the notion of evaluation of a formula is the definition of negation, it arises as a generalization of Keisler's definition for which a condition p forces the negation of a formula $\neg\varphi$ if, and only if, φ is not forced by any condition greater or equal then p. Notice also that, in the definition of the forcing value of a formula of the form $\psi \to \chi$, the infimum effects only the antecedent because ψ implicitly appears negatively in the formula.

On the one hand such a requirement for the forcing value of the negation of a formula guarantees the "coherence" of the notion of forcing. On the other hand this requirement allows to define a kind of truth degree of formulas by steps.

In next lemmas are summarized some basic properties of forcing which make more explicit the difference w.r.t. the notion of evaluation.

LEMMA 3.3. *For any forcing property, any condition p in it and any sentence φ, ψ and $\forall x \chi(x)$ of $\mathcal{L}(C)$ we have :*

1.
$$[\neg\neg\varphi]_p = \bigwedge_{p \leq q} \bigvee_{q \leq v} [\varphi]_v;$$

2.

$$[\varphi \to \psi]_p = [\neg\varphi]_p \oplus [\psi]_p;$$

3.

$$[\varphi \underline{\oplus} \psi]_p = [\neg\neg\varphi]_p \oplus [\psi]_p;$$

4.

$$[\forall x \chi(x)]_p = \bigwedge_{p \le q} \bigwedge_{c \in C} \bigvee_{q \le r} [\chi(c)]_r.$$

PROOF. 1. Obvious.

2.

$$[\varphi \to \psi]_p = \bigwedge_{p \le q} ([\varphi]_q \Rightarrow [\psi]_p) = \bigwedge_{p \le q} \left([\varphi]_q^* \oplus [\psi]_p\right)$$
$$= \left(\bigwedge_{p \le q} [\varphi]_q^*\right) \oplus [\varphi]_p = [\neg\varphi]_p \oplus [\psi]_q.$$

3. According to 2 we have

$$[\varphi \underline{\oplus} \psi]_p = [\neg\varphi \to \psi]_p = [\neg\neg\varphi] \oplus [\psi]_p.$$

4. Obvious.

■

Forcing can be seen then as the process of defining a "truth value" of a formula by steps. Next lemma shows that such an assignment is weakly increasing.

LEMMA 3.4. *For all $p \le q$ and $\varphi \in E$ we have $[\varphi]_p \le [\varphi]_q$.*

PROOF. By induction on the complexity of φ:

- if φ is atomic then $[\varphi]_p = f(p, \varphi) \le f(q, \varphi) = [\varphi]_q$.
- if $\varphi = \neg\psi$ then

$$[\varphi]_p = \bigwedge_{p \le r} [\psi]_r^* \le \bigwedge_{q \le r} [\psi]_r^* = [\varphi]_q.$$

- Assume $\varphi = \psi \to \chi$. Using the induction hypothesis, $[\chi]_p \le [\chi]_q$, so $[\psi]_r \Rightarrow [\chi]_p \le [\psi]_r \Rightarrow [\chi]_q$ for all conditions r. Hence

$$[\varphi]_p = \bigwedge_{p \le r} ([\varphi]_r \Rightarrow [\chi]_p) \le \bigwedge_{q \le r} ([\psi]_r \Rightarrow [\chi]_q) = [\varphi]_q.$$

- Assume $\varphi = \exists x \psi(x)$, by the induction hypothesis $[\psi(c)]_p \leq [\psi(c)]_q$ for all $c \in C$. Therefore

$$[\varphi]_p = \bigvee_{c \in C} [\psi(c)]_p \leq \bigvee_{c \in C} [\psi(c)]_q = [\varphi]_q.$$

■

LEMMA 3.5. *For any $p \in P$ and $\varphi \in E$, we have*

1. $[\varphi]_p \leq [\neg\neg\varphi]_p$.
2. $[\neg\varphi]_p = [\neg\neg\neg\varphi]_p$.

PROOF. 1. For any $q \geq p$,

$$[\varphi]_p \leq [\varphi]_q \leq \bigvee_{q \leq r} [\varphi]_r,$$

therefore

$$[\varphi]_p \leq \bigwedge_{p \leq q} \bigvee_{q \leq r} [\varphi]_r = [\neg\neg\varphi]_p.$$

2. We remark that

$$[\neg\neg\neg\varphi]_p = \bigwedge_{q \geq p} \bigvee_{r \geq q} \bigwedge_{s \geq r} [\varphi]_s^*.$$

Let $q \geq p$. For any $r \geq q$ we have $[\varphi]_q \leq [\varphi]_r$, therefore

$$\bigwedge_{s \geq r} [\varphi]_s^* \leq [\varphi]_r^* \leq [\varphi]_q^*.$$

Thus, for any $q \geq p$, we have

$$\bigvee_{r \geq q} \bigwedge_{s \geq r} [\varphi]_s^* \leq [\varphi]_q^*,$$

hence

$$[\neg\neg\neg\varphi]_p \leq \bigwedge_{q \geq p} [\varphi]_q^* = [\neg\varphi]_p.$$

The converse inequality $[\neg\varphi]_p \leq [\neg\neg\neg\varphi]_p$ holds by the previous item.

■

We are ready now to give a definition which will allow us to establish a link between forcing value and truth value of a formula. The forcing value of a formula at some condition can be seen as a partial piece of information, the following notion formalizes a way to complete this information.

Definition 3.6. A non empty subset G of P is called *generic* if the following conditions hold

(i) If $p \in G$ and $q \leq p$ then $q \in G$,

(ii) For any $p, g \in G$ there exists $v \in G$ such that $p, g \leq v$;

(iii) For any $\varphi \in E$ there exists $p \in G$ such that $[\varphi]_p \oplus [\neg\varphi]_p = 1$.

Definition 3.7. Given a forcing property $\langle P, \leq, 0, f \rangle$, a model $\mathfrak{A}$ is *generated by* a generic set G if for all $\varphi \in E$ and $p \in G$ we have $[\varphi]_p \leq \|\varphi\|_{\mathfrak{A}}$.

A model $\mathfrak{A}$ is *generic* for $p \in P$ if it is generated by a generic subset G which contains p. $\mathfrak{A}$ is generic if it is generic for 0.

Although we gave the definition of a generic model for 0, we will not explore here such a class of structures. Nevertheless it has, in the classical case, peculiar and notable properties which might me extended to the many-valued case. We leave this as an open problem now, in order to have space to establish some more general results.

The notion of genericity is a bridge between the notions of forcing value and truth value. Next theorem give a way to establish this link for any forcing property in a canonical way.

Theorem 3.8 (Generic Model Theorem). *Let $\langle P, \leq, 0, f \rangle$ be a forcing property and $p \in P$. Then there exists a generic model for p.*

Proof. The proof is based on the following two lemmas (Lemmas 3.9 and 3.10). ■

Lemma 3.9. *For any $p \in P$ there exists a generic set G such that $p \in G$.*

Proof. The set E is countable, then one can consider an enumeration $\varphi_0, \varphi_1 ... \varphi_n, ...$ of its elements. By induction, we shall construct a sequence of conditions $p = p_0 \leq p_1 \leq ... \leq p_n \leq ...$ such that $[\varphi_n]_{p_{n+1}} \oplus [\neg\varphi_n]_{p_{n+1}} = 1$ for all $n \in \omega$. Assume that there are $p_0 \leq p_1 \leq ... \leq p_n$ with this property. In order to define p_{n+1} we shall consider the following two cases:

(a) $[\varphi_n]_{p_n} \oplus [\neg\varphi_n]_{p_n} = 1$. We set $p_{n+1} = p_n$.

(b) $[\varphi_n]_{p_n} \oplus [\neg\varphi_n]_{p_n} < 1$. By absurdum, we assume that

$$[\varphi_n]_q \oplus [\neg\varphi_n]_q < 1 \text{ for all } q \geq p_n. \qquad \text{(Abs)}$$

Let us consider a cardinal $k > 2^\omega$. Using (Abs) we shall define, by transfinite induction, an increasing sequence of conditions $(p_{n_\alpha})_{\alpha<k}$ such that

$$\alpha < \beta < k \Longrightarrow [\varphi_n]_{p_{n_\alpha}} < [\varphi_n]_{p_{n_\beta}}.$$

We shall distinguish the following three cases:

- $\alpha = 0$. We set $p_{n_0} = p_n$.
- $\alpha = \beta + 1$ is a successor ordinal. Since the properties (Abs) holds for p_{n_β}, we have $[\varphi_n]_{p_{n_\beta}} \oplus [\neg\varphi_n]_{p_{n_\beta}} < 1$, hence

$$\bigwedge_{p_{n_\beta} \leq q} \left([\varphi_n]_{p_{n_\beta}} \oplus [\varphi_n]_q^*\right) = [\varphi_n]_{p_{n_\beta}} \oplus \bigwedge_{p_{n_\beta} \leq q} [\varphi_n]_q^* < 1.$$

  Then there exists a condition $g \geq p_{n_\beta}$ such that, by Lemma 3.3 (b)

$$[\varphi_n]_g \Rightarrow [\varphi_n]_{p_{n_\beta}} = [\varphi]_g^* \oplus [\varphi_n]_{p_{n_\beta}} < 1,$$

  hence $[\varphi_n]_g \not\leq [\varphi_n]_{p_{n_\beta}}$, hence $[\varphi_n]_{p_{n_\beta}} < [\varphi_n]_g$.
  We define $p_{n_\alpha} = g$.
- α is a limit ordinal. By construction, $\{p_{n_\beta} \mid \beta < \alpha\}$ is ordinal-indexed, so, by Definition 3.1, (ii), it has an upper bound q. We define $p_{n_\alpha} = q$. Let $\beta < \alpha$, hence, according to Lemma 3.4, we have $[\varphi_n]_{p_{n_\beta}} \leq [\varphi_n]_{p_{n_\alpha}}$. We must prove then $[\varphi_n]_{p_{n_\beta}} \neq [\varphi_n]_{p_{n_\alpha}}$. Fix β and consider $\gamma = \beta + 1 < \alpha$ then $[\varphi_n]_{p_{n_\beta}} < [\varphi_n]_{p_{n_\gamma}} \leq [\varphi_n]_{p_{n_\alpha}} = [\varphi_n]_{p_{n_\beta}}$.
  We have obtained a contradiction, then $[\varphi_n]_{p_{n_\beta}} < [\varphi_n]_{p_{n_\alpha}}$.

In this way, the construction of the sequence $(p_{n_\alpha})_{\alpha<k}$ is complete. But then $k \leq 2^\omega$ and this contradicts our choice of k. This contradiction shows that (Abs) fails, i.e. $[\varphi_n]_q \oplus [\neg\varphi_n]_q = 1$ for some $q \geq p_n$. We set $p_{n+1} = q$. Then we have proved the existence of the sequence $(p_n)_{n\in\omega}$. If we denote

$$G = \{q \in P \mid q \leq p_n \text{ for some } n \in \omega\},$$

then it is easy to see that G is generic for p. ■

LEMMA 3.10. *Every generic set G generates a denumerable model.*

PROOF. For any $\varphi \in E$ we denote

$$T(\varphi) = \bigvee_{p\in G} [\varphi]_p.$$

We shall define a structure whose universe M is the set of all constants of the language $\mathcal{L}(C)$ (recall that C is a denumerable set of new constants).
If $P(x_1, ..., x_n)$ is an atomic formula of $\mathcal{L}(C)$ and $c_1, ..., c_n \in M$ then we define $P^{\mathfrak{M}}(c_1, ..., c_n) = T(P(c_1, ..., c_n))$. For any constant c of $\mathcal{L}(C)$ we take $c^{\mathfrak{M}} = c$. The function $T : E \to [0, 1]$ has the following properties:

1. $T(\varphi) \oplus T(\neg\varphi) = 1$, for all $\varphi \in E$;
2. $T(\varphi) \odot T(\neg\varphi) = 0$, for all $\varphi \in E$;
3. $T(\neg\varphi) = T(\varphi)^*$, for all $\varphi \in E$;
4. $T(\varphi \rightarrow \psi) = T(\varphi) \Rightarrow T(\psi)$, for all $\varphi, \psi \in E$;
5. $T(\varphi \underline{\oplus} \psi) = T(\varphi) \oplus T(\psi)$, for all $\varphi, \psi \in E$;
6. $T(\exists x\varphi(x)) = \bigvee_{c\in C} T(\varphi(c))$, for all sentences of the form $\exists x\varphi(x)$.

Now we shall prove points 1 to 6.

1. We fix $q_o \in P$. Then

$$\begin{aligned} T(\varphi) \oplus T(\neg\varphi) &= (\bigvee_{p\in G} [\varphi]_p) \oplus (\bigvee_{q\in G} [\neg\varphi]_q) \\ &= (\bigvee_{p\in G} [\varphi]_p) \oplus (\bigvee_{q\in G} \bigwedge_{q\leq r} [\varphi]^*_r) \\ &\geq (\bigvee_{p\in G} [\varphi]_p) \oplus (\bigwedge_{q_0\leq r} [\varphi]^*_r) \\ &= \bigwedge_{q_0\leq r} ((\bigvee_{p\in G} [\varphi]_p) \oplus [\varphi]^*_r) \\ &\geq \bigwedge_{q_0\leq r} ([\varphi]_r \oplus [\varphi]^*_r) = 1. \end{aligned}$$

2.

$$\begin{aligned} T(\varphi) \odot T(\neg\varphi) &= (\bigvee_{p\in G} [\varphi]_p) \odot (\bigvee_{q\in G} [\neg\varphi]_q]) \\ &= \bigvee_{p,q\in G} ([\varphi]_p \odot \bigwedge_{q\leq r} [\varphi]^*_r) \\ &= \bigvee_{p,q\in G} \bigwedge_{q\leq r} ([\varphi]_p \odot [\varphi]^*_r). \end{aligned}$$

For any $p, q \in G$ there exists $r \in G$ such that $p, q \leq r$, so $[\varphi]_p \leq [\varphi]_r$, therefore $[\varphi]_p \odot [\varphi]^*_r \leq [\varphi]_p \odot [\varphi]^*_p = 0$. Thus we obtained $T(\varphi) \odot T(\neg\varphi) = 0$.

3. By 1 and 2.

4. We have

$$
\begin{aligned}
&T(\varphi \to \psi) = \bigvee_{p \in G} [\varphi \to \psi]_p = \bigvee_{p \in G} ([\neg\varphi]_p \oplus [\psi]_p);\\
&T(\varphi) \Rightarrow T(\psi) = T(\varphi)^* \oplus T(\psi) = T(\neg\varphi) \oplus T(\psi)\\
&= \bigvee_{q,r \in G} ([\neg\varphi]_q \oplus [\psi]_r).
\end{aligned}
$$

   Thus $T(\varphi \to \psi) \leq T(\varphi) \Rightarrow T(\psi)$. Let $q, r, \in G$ then $q, r \leq s$ for some $s \in G$. Then $[\neg\varphi]_q \oplus [\psi]_r \leq [\neg\varphi]_s \oplus [\psi]_s \leq T(\varphi \to \psi)$. This inequality holds for any $q, r \in G$, hence $T(\varphi) \Rightarrow T(\psi) \leq T(\varphi \to \psi)$.

5. By 3 and 4 we have

$$
\begin{aligned}
T(\varphi \underline{\oplus} \psi) &= T(\neg\varphi \to \psi) = T(\neg\varphi) \Rightarrow T(\psi)\\
&= T(\varphi)^* \Rightarrow T(\psi) = T(\varphi) \oplus T(\psi).
\end{aligned}
$$

6.

$$
\begin{aligned}
T(\exists x\varphi(x)) &= \bigvee_{p \in G} [\exists x\varphi(x)]_p = \bigvee_{p \in G} \bigvee_{c \in C} [\varphi(c)]_p\\
&= \bigvee_{c \in C} \bigvee_{p \in G} [\varphi(c)]_p = \bigvee_{c \in C} T(\varphi(c)).
\end{aligned}
$$

From 1-6 we get $T(\varphi) = \|\varphi\|_{\mathfrak{M}}$, for any $\varphi \in E$. Thus $[\varphi]_p \leq \|\varphi\|_{\mathfrak{M}}$, for all $\varphi \in E$ and $p \in G$, hence $\mathfrak{M}$ is generated by G. ■

An easy inspection on the proof of the previous theorem shows that the following corollary holds.

Corollary 3.11. *If p belongs to some generic set G which has a maximum g, then there exists $\mathfrak{M}$, generic model for p, such that $[\varphi]_g = \|\varphi\|_{\mathfrak{M}}$*

Corollary 3.12. *For any $\varphi \in E$ and $p \in P$ we have*

$$
(\sharp) \qquad [\neg\neg\varphi]_p = \bigwedge\{\|\varphi\|_{\mathfrak{M}} \mid \mathfrak{M} \textit{ is a generic structure for } p\}.
$$

Proof. Let us denote by a the right member of $(\sharp)$. Let $\mathfrak{M}$ be a generic structure for p (by Theorem 3.8). Then, by definition, $[\neg\neg\varphi]_p \leq \|\neg\neg\varphi\|_{\mathfrak{M}} = \|\varphi\|_{\mathfrak{M}}$, hence $[\neg\neg\varphi]_p \leq a$.

Now we shall prove that $a \leq [\neg\neg\varphi]_p$. Let $g \geq p$. By Theorem 3.8, there exists a structure $\mathfrak{M}$, generic for g, since p belongs to every generic to which g belongs, $\mathfrak{M}$ is also generic for p. Thus $[\neg\varphi]_g \leq \|\neg\varphi\|_{\mathfrak{M}}$, hence $\|\varphi\|_{\mathfrak{M}} = \left\|\neg\varphi\right\|^*_{\mathfrak{M}} \leq \left[\neg\varphi\right]^*_g$ for all $g \geq p$. Therefore

$$a \leq \|\varphi\|_{\mathfrak{M}} \leq \bigwedge_{p \leq g} \left[\neg\varphi\right]^*_g = [\neg\neg\varphi]_p.$$

■

Notice that both corollaries above give a sharp relation between forcing value and truth value of a formula. This is not the case in Theorem 3.8 in which the link is given only by an inequality.

4. Infinite Forcing and General Properties

In this section we deal with infinite forcing for models of Łukasiewicz predicate logic. Infinite forcing can be seen as a variation of finite forcing which is tied to a given class of structures. It will be seen that in many cases infinite forcing behaves similarly to the standard semantic, but they do not coincide.

Throughout this section we work with a fixed but arbitrary Ł∀-language $\mathcal{L}$, again the word *structure* will be used to refer to a standard structure of $\mathcal{L}$, but the results can be applied to any structure, once the definition of forcing is generalized to take values in an arbitrary complete MV-algebra. For any structure $\mathfrak{A}$, we shall denote by $\mathcal{L}(\mathfrak{A})$ the language obtained from $\mathcal{L}$ by adding the elements of the universe A of $\mathfrak{A}$, as new constants.

We give now some basic generalizations of classical concepts of model theory.

Definition 4.1.

- Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures. $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$ (or $\mathfrak{B}$ is an *extension* of $\mathfrak{A}$) if $A \subseteq B$, and for any atomic sentence φ of $\mathcal{L}(\mathfrak{A})$ we have $\|\varphi\|_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{B}}$. By $\mathfrak{A} \subseteq \mathfrak{B}$ we mean that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$.
- $\mathfrak{A}$ is a *elementary substructure* of $\mathfrak{B}$ (or $\mathfrak{B}$ is an *elementary extension* of $\mathfrak{A}$) if $A \subseteq B$ and for any sentence φ of $\mathcal{L}(\mathfrak{A})$ we have $\|\varphi\|_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{B}}$ (in this case we denote $\mathfrak{A} \preccurlyeq \mathfrak{B}$).

A class Σ of structures is *model-complete* if for any structures $\mathfrak{A}, \mathfrak{B} \in \Sigma$, $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{A} \preccurlyeq \mathfrak{B}$.

Let Σ_1, Σ_2 be two classes of structures. Σ_1 is *model-consistent* with Σ_2 if $\Sigma_1 \subseteq \Sigma_2$ and for any $\mathfrak{A}_2 \in \Sigma_2$ there exists $\mathfrak{A}_1 \in \Sigma_1$ such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$.

An *existential sentence* φ of $\mathcal{L}$ has the form $\exists x_1 ... \exists x_n \varphi(x_1, ..., x_n)$, where $\varphi(x_1, ..., x_n)$ is a quantifier-free formula of $\mathcal{L}$.

Let α be a cardinal and $\langle \mathfrak{A}_\xi : \xi < \alpha \rangle$ a chain of structures, i.e., $\mathfrak{A}_\xi \subseteq \mathfrak{A}_\eta$ for all ordinals ξ, η such that $\xi < \eta < \alpha$. Now we shall define the union $\mathfrak{A} = \bigcup_{\xi<\alpha} \mathfrak{A}_\xi$ of this chain. Let us denote $A = \bigcup_{\xi<\alpha} A_\xi$. Let $\varphi(x_1, ..., x_n)$ be an atomic formula $\mathcal{L}$ and $a_1, ..., a_n \in A$, hence $a_1, ..., a_n \in A_\xi$ for some ordinal $\xi < \alpha$. Then we set $\|\varphi(a_1, ..., a_n)\|_{\mathfrak{A}} = \|\varphi(a_1, ..., a_n)\|_{\mathfrak{A}_\xi}$. It is easy to see that $\|\varphi(a_1, ..., a_n)\|_{\mathfrak{A}}$ is well defined. We observe that if c is a constant of $\mathcal{L}$, then $c^{\mathfrak{A}} = c^{\mathfrak{A}_\xi}$ for any $\xi < \alpha$.

A class Σ of structures is *inductive* if it closed under the unions of chains of structures.

Let $\langle \mathfrak{A}_\xi : \xi < \alpha \rangle$ be a chain of structures. We say that $\langle \mathfrak{A}_\xi : \xi < \alpha \rangle$ is an *elementary chain* if $\mathfrak{A}_\xi \preccurlyeq \mathfrak{A}_\eta$ for all ordinals $\xi < \eta < \alpha$.

LEMMA 4.2. *Let $\langle \mathfrak{A}_\xi : \xi < \alpha \rangle$ be an elementary chain and $\mathfrak{A} = \bigcup_{\xi<\alpha} \mathfrak{A}_\xi$. Then $\mathfrak{A}_\xi \preccurlyeq \mathfrak{A}$ for all $\xi < \alpha$.*

PROOF. We shall prove, by induction on the complexity of formulas, that for all $\xi < \alpha$ and for all sentences φ of $\mathcal{L}(\mathfrak{A}_\xi)$ we have $\|\varphi\|_{\mathfrak{A}_\xi} = \|\varphi\|_{\mathfrak{A}}$.

We shall treat only the case $\varphi = \exists x \psi(x)$. Let $\xi < \alpha$. By induction hypothesis, $\|\psi(a)\|_{\mathfrak{A}_\xi} = \|\psi(a)\|_{\mathfrak{A}}$ for any $a \in A_\xi$.

Let $b \in A$ then there exists $\eta < \alpha$ such that $b \in A_\eta$. Suppose that $\xi < \eta$, hence

$$\|\psi(b)\|_{\mathfrak{A}} = \|\psi(b)\|_{\mathfrak{A}_\eta} \leq \|\exists x \psi(x)\|_{\mathfrak{A}_\eta} = \|\exists x \psi(x)\|_{\mathfrak{A}_\xi}.$$

Where the last equality holds because $\mathfrak{A}_\xi \preccurlyeq \mathfrak{A}_\eta$. This inequality holds for any $b \in A$, therefore

$$\|\exists x \psi(x)\|_{\mathfrak{A}} = \bigvee_{b \in A} \|\psi(b)\|_{\mathfrak{A}} \leq \|\exists x \psi(x)\|_{\mathfrak{A}_\xi}.$$

The converse inequality follows immediately:

$$\|\exists x \psi(x)\|_{\mathfrak{A}_\xi} = \bigvee_{b \in A_\xi} \|\psi(b)\|_{\mathfrak{A}_\xi} \leq \bigvee_{b \in A} \|\psi(b)\|_{\mathfrak{A}} = \|\exists x \psi(x)\|_{\mathfrak{A}}.$$

■

In what follows all structures will be assumed to be members of a fixed, but arbitrary, inductive class Σ of structures in a common language $\mathcal{L}$.

DEFINITION 4.3. For any structure $\mathfrak{A} \in \Sigma$ and for any sentence φ of $\mathcal{L}(\mathfrak{A})$ we shall define by induction the real number $[\varphi]^{\Sigma}_{\mathfrak{A}} \in [0, 1]$:

1. If φ is an atomic sentence then

$$[\varphi]^{\Sigma}_{\mathfrak{A}} = ||\varphi||_{\mathfrak{A}};$$

2. If $\varphi = \neg\psi$ then
$$[\varphi]^{\Sigma}_{\mathfrak{A}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} [\psi]^{*}_{\mathfrak{B}};$$
3. If $\varphi = \psi \to \chi$ then
$$[\varphi]^{\Sigma}_{\mathfrak{A}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} ([\psi]_{\mathfrak{B}} \Rightarrow [\chi]_{\mathfrak{A}});$$
4. If $\varphi = \exists x\psi(x)$ then
$$[\varphi]^{\Sigma}_{\mathfrak{A}} = \bigvee_{a\in\mathfrak{A}} [\psi(a)]_{\mathfrak{A}}.$$

$[\varphi]^{\Sigma}_{\mathfrak{A}}$ will be called the *forcing value* of φ in $\mathfrak{A}$. We define $\mathfrak{A} \Vdash \varphi$ ($\mathfrak{A}$ *forces* φ) if $[\varphi]^{\Sigma}_{\mathfrak{A}} = 1$.

Notice that the forcing value of a formula depends on the class Σ that we are considering and this motivates the upper-script in our definition, nevertheless we fixed an arbitrary class Σ and we will henceforth drop the upper-script.

As finite forcing, also infinite forcing can be viewed as another kind of semantic. The main difference consists in the behavior w.r.t. the negation. Note however that as far as φ is a quantifier-free formula $[\varphi]_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{A}}$.

A natural question is whether $[\varphi]_{\mathfrak{A}} = 1$ for any formal theorem φ of Ł∀. The following example shows that the answer is negative:

Example 4.4. Let us consider a language $\mathcal{L}'$ of Ł∀ with a unique unary predicate symbol R. We define two standard structures $\mathfrak{A}$ and $\mathfrak{B}$ by putting
$$\begin{array}{ll} \mathfrak{A} = \{a, b\}, & R^{\mathfrak{A}}(a) = 1/2, \quad R^{\mathfrak{A}}(b) = 1/3 \\ \mathfrak{B} = \{a, b, c\}, & R^{\mathfrak{B}}(a) = 1/2, \quad R^{\mathfrak{B}}(b) = 1/3, \quad R^{\mathfrak{B}}(c) = 1. \end{array}$$
Of course $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. Let us take $\Sigma = \{\mathfrak{A}, \mathfrak{B}\}$ and consider the following sentence of $\mathcal{L}'$
$$\exists x R(x) \to \exists x R(x).$$
This sentence is a formal theorem of Ł∀ (identity principle). We remark that
$$[\exists x R(x)]_{\mathfrak{A}} = [R(a)]_{\mathfrak{A}} \vee [R(b)]_{\mathfrak{A}} = \max(1/2, 1/3) = 1/2$$
$$[\exists x R(x)]_{\mathfrak{B}} = [R(a)]_{\mathfrak{B}} \vee [R(b)]_{\mathfrak{B}} \vee [R(c)]_{\mathfrak{B}} = \max(1/2, 1/3, 1) = 1.$$
Applying Definition 4.3 we get
$$[\exists x R(x) \to \exists x R(x)]_{\mathfrak{A}} = [\exists x R(x)]_{\mathfrak{B}} \Rightarrow [\exists x R(x)]_{\mathfrak{A}} = 1 \to 1/2 = 1/2.$$

The following lemmas show some properties which infinite forcing has in common with finite forcing.

LEMMA 4.5. *For any structure $\mathfrak{A}$ and for any sentences φ, ψ and $\forall x\chi(x)$ of $\mathcal{L}(\mathfrak{A})$ the following equalities hold:*

1.
$$[\neg\neg\varphi]_{\mathfrak{A}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} \bigvee_{\mathfrak{B}\subseteq\mathfrak{C}} [\varphi]_{\mathfrak{C}};$$

2.
$$[\varphi \to \psi]_{\mathfrak{A}} = [\neg\varphi]_{\mathfrak{A}} \oplus [\psi]_{\mathfrak{A}};$$

3.
$$[\varphi\underline{\oplus}\psi]_{\mathfrak{A}} = [\neg\neg\varphi]_{\mathfrak{A}} \oplus [\psi]_{\mathfrak{A}};$$

4.
$$[\forall x\chi(x)]_{\mathfrak{A}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} \bigwedge_{b\in\mathfrak{B}} \bigvee_{\mathfrak{B}\subseteq\mathfrak{C}} [\chi(b)]_{\mathfrak{C}}.$$

PROOF. 1.

$$[\neg\neg\varphi]_{\mathfrak{A}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} [\neg\varphi]^*_{\mathfrak{B}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} \Big(\bigwedge_{\mathfrak{B}\subseteq\mathfrak{C}} [\varphi]^*_{\mathfrak{C}}\Big)^* = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} \bigvee_{\mathfrak{B}\subseteq\mathfrak{C}} [\varphi]_{\mathfrak{C}}.$$

2.

$$\begin{aligned}[\varphi \to \psi]_{\mathfrak{A}} &= \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} ([\varphi]_{\mathfrak{B}} \Rightarrow [\psi]_{\mathfrak{A}}) = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} ([\varphi]^*_{\mathfrak{B}} \oplus [\psi]_{\mathfrak{A}}) \\ &= \Big(\bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} [\varphi]^*_{\mathfrak{B}}\Big) \oplus [\psi]_{\mathfrak{A}} = [\neg\varphi]_{\mathfrak{A}} \oplus [\psi]_{\mathfrak{A}}.\end{aligned}$$

3. According to 2 we have:

$$[\varphi\underline{\oplus}\psi]_{\mathfrak{A}} = [\neg\varphi \to \psi]_{\mathfrak{A}} = [\neg\neg\varphi]_{\mathfrak{A}} \oplus [\psi]_{\mathfrak{A}}.$$

4. Similarly.

■

LEMMA 4.6. *If $\mathfrak{A} \subseteq \mathfrak{B}$ and φ is a sentence of $\mathcal{L}(\mathfrak{A})$ then $[\varphi]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{B}}$.*

PROOF.

- If φ is atomic then $[\varphi]_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{B}} = [\varphi]_{\mathfrak{B}}$.

- If $\varphi = \neg\psi$ then

$$[\varphi]_{\mathfrak{A}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{C}} [\psi]^*_{\mathfrak{C}} \leq \bigwedge_{\mathfrak{B}\subseteq\mathfrak{C}} [\psi]^*_{\mathfrak{C}} = [\varphi]_{\mathfrak{B}}.$$

- Assume $\varphi = \psi \to \chi$. By the induction hypothesis, $[\chi]_{\mathfrak{A}} \leq [\chi]_{\mathfrak{B}}$, therefore $[\psi]_{\mathfrak{C}} \Rightarrow [\chi]_{\mathfrak{A}} \leq [\psi]_{\mathfrak{C}} \Rightarrow [\chi]_{\mathfrak{B}}$ for all structures $\mathfrak{C}$. Then

$$[\varphi]_{\mathfrak{A}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{C}} \big([\psi]_{\mathfrak{C}} \Rightarrow [\chi]_{\mathfrak{A}}\big) \leq \bigwedge_{\mathfrak{B}\subseteq\mathfrak{C}} \big([\psi]_{\mathfrak{C}} \Rightarrow [\chi]_{\mathfrak{B}}\big) = [\varphi]_{\mathfrak{B}}.$$

- Assume $\varphi = \exists x\psi(x)$ and $[\psi(a)]_{\mathfrak{A}} \leq [\psi(a)]_{\mathfrak{B}}$ for any $a \in \mathfrak{A}$. Then

$$[\varphi]_{\mathfrak{A}} = \bigvee_{a\in\mathfrak{A}} [\psi(a)]_{\mathfrak{A}} \leq \bigvee_{b\in\mathfrak{B}} [\psi(b)]_{\mathfrak{B}} = [\varphi]_{\mathfrak{B}}.$$

■

The following lemma shows how the interpretation of formulas given by infinite forcing has a behavior which is less strict on negative formulas, in a way similar to the one of intuitionistic logic.

LEMMA 4.7. *For any formula φ and $\mathfrak{A} \in \Sigma$:*

1. $[\varphi]_{\mathfrak{A}} \odot [\neg\varphi]_{\mathfrak{A}} = 0$.
2. $[\varphi]_{\mathfrak{A}} \leq [\neg\neg\varphi]_{\mathfrak{A}}$.
3. $[\neg\varphi]_{\mathfrak{A}} = [\neg\neg\neg\varphi]_{\mathfrak{A}}$.

PROOF. 1. According to

$$[\neg\varphi]_{\mathfrak{A}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} [\varphi]^*_{\mathfrak{B}} \leq [\varphi]^*_{\mathfrak{A}}.$$

we get $[\varphi]_{\mathfrak{A}} \odot [\neg\varphi]_{\mathfrak{A}} = 0$.

2. For any structure $\mathfrak{B}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ we have

$$[\varphi]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{B}} \leq \bigvee_{\mathfrak{B}\subseteq\mathfrak{C}} [\varphi]_{\mathfrak{C}},$$

hence

$$[\varphi]_{\mathfrak{A}} \leq \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} \bigvee_{\mathfrak{B}\subseteq\mathfrak{C}} [\varphi]_{\mathfrak{C}} = [\neg\neg\varphi]_{\mathfrak{A}}.$$

3. By Lemma 4.5, 1, we have

$$[\neg\neg\neg\varphi]_{\mathfrak{A}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} \bigvee_{\mathfrak{B}\subseteq\mathfrak{C}} [\neg\varphi]_{\mathfrak{C}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} \bigvee_{\mathfrak{B}\subseteq\mathfrak{C}} \bigwedge_{\mathfrak{C}\subseteq\mathfrak{D}} [\varphi]^*_{\mathfrak{D}}.$$

Let $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$. By Lemma 4.6, $[\varphi]_{\mathfrak{B}} \leq [\varphi]_{\mathfrak{C}}$, therefore

$$\bigvee_{\mathfrak{C}\subseteq\mathfrak{D}} [\varphi]^*_{\mathfrak{D}} \leq [\varphi]^*_{\mathfrak{C}} \leq [\varphi]^*_{\mathfrak{B}}.$$

Hence for all structures $\mathfrak{B}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ we get

$$\bigwedge_{\mathfrak{B}\subseteq\mathfrak{C}} \bigvee_{\mathfrak{C}\subseteq\mathfrak{D}} [\varphi]^*_{\mathfrak{D}} \leq [\varphi]^*_{\mathfrak{B}}.$$

Thus

$$[\neg\neg\neg\varphi]_{\mathfrak{A}} \leq \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} [\varphi]^*_{\mathfrak{B}} = [\neg\varphi]_{\mathfrak{A}}.$$

The converse inequality $[\neg\varphi]_{\mathfrak{A}} \leq [\neg\neg\neg\varphi]_{\mathfrak{A}}$ follows by the previous item.

■

PROPOSITION 4.8.

1. $[\exists x\exists y\varphi(x,y)]_{\mathfrak{A}} = [\exists y\exists x\varphi(x,y)]_{\mathfrak{A}}$*;*

2. $[\exists x\forall y\varphi(x,y)]_{\mathfrak{A}} \leq [\forall y\exists x\varphi(x,y)]_{\mathfrak{A}}$*.*

PROOF. 1. The equality follows directly from the clause 4 of Definition 4.3.
2. Let $\mathfrak{B}$ an extension of $\mathfrak{A}$, $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Then

$$\begin{aligned}
[\forall x\varphi(a,y)]_{\mathfrak{A}} &= \bigwedge_{\mathfrak{A}\subseteq\mathfrak{D}} \bigwedge_{d\in\mathfrak{D}} \bigvee_{\mathfrak{D}\subseteq\mathfrak{E}} [\varphi(a,d)]_{\mathfrak{E}} \\
&\leq \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} ([\varphi(a,b)]_{\mathfrak{C}}) \\
&\leq \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} \bigvee_{c\in\mathfrak{C}} ([\varphi(a,c)]_{\mathfrak{C}}) \\
&= \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} ([\exists x\varphi(x,b)]_{\mathfrak{C}}).
\end{aligned}$$

Therefore

$$[\exists x\forall y\varphi(x,y)]_{\mathfrak{A}} = \bigvee_{a\in\mathfrak{A}} ([\forall y\varphi(a,y)]_{\mathfrak{A}}) \leq \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} ([\exists x\varphi(x,b)]_{\mathfrak{C}}).$$

This last inequality holds for all extensions $\mathfrak{B}$ of $\mathfrak{A}$ and $b \in \mathfrak{B}$, hence

$$[\exists x \forall y \varphi(x,y)]_{\mathfrak{A}} \leq \bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} \bigwedge_{b\in\mathfrak{B}} \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} [\exists x \varphi(x,b)]_{\mathfrak{C}} = [\forall x \exists y \varphi(x,y)]_{\mathfrak{A}}.$$

■

PROPOSITION 4.9. *Given two formulas of $\mathcal{L}$, φ and $\psi(x)$, suppose that x is not free in φ, then*

1. $[\forall x(\varphi \to \psi(x))]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{A}} \Rightarrow [\forall x \psi(x)]_{\mathfrak{A}}$;
2. $[\exists x(\varphi \to \psi(x)]_{\mathfrak{A}} = [\varphi]_{\mathfrak{A}} \Rightarrow [\exists x \psi(x)]_{\mathfrak{A}}$.

PROOF. 1. Let $\mathfrak{B}$ an extension of $\mathfrak{A}$, $\mathfrak{C}$ an extension of $\mathfrak{B}$ in Σ and $b \in \mathfrak{B}$. By Lemmas 4.6 and 6.1:

$$\begin{aligned}[\varphi]_{\mathfrak{A}} \odot [\varphi \to \psi(x)]_{\mathfrak{C}} &\leq [\varphi]_{\mathfrak{C}} \odot [\varphi \to \psi(x)]_{\mathfrak{C}} \\ &\leq [\varphi]_{\mathfrak{C}} \odot ([\varphi]_{\mathfrak{C}} \to [\psi]_{\mathfrak{C}}) \\ &= [\varphi]_{\mathfrak{C}} \wedge [\psi]_{\mathfrak{C}} \leq [\psi]_{\mathfrak{C}},\end{aligned}$$

therefore

$$[\varphi]_{\mathfrak{A}} \odot \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} [\varphi \to \psi(b)]_{\mathfrak{C}} = \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} ([\varphi]_{\mathfrak{A}} \odot [\varphi \to \psi(b)]_{\mathfrak{C}}) \leq \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} [\psi(b)]_{\mathfrak{C}}.$$

This is true for all extensions $\mathfrak{B}$ of $\mathfrak{A}$ and $b \in \mathfrak{B}$, hence

$$\begin{aligned}[\varphi]_{\mathfrak{A}} \odot [\forall x(\varphi \to \psi(x))]_{\mathfrak{A}} &= [\varphi]_{\mathfrak{A}} \odot \bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} \bigwedge_{b\in\mathfrak{B}} \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} [\varphi \to \psi(b)]_{\mathfrak{C}} \\ &= \bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} \bigwedge_{b\in\mathfrak{B}} \left([\varphi]_{\mathfrak{A}} \odot \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} [\varphi \to \psi(b)]_{\mathfrak{C}} \right) \\ &\leq \bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} \bigwedge_{b\in\mathfrak{B}} \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} [\psi(b)]_{\mathfrak{C}}.\end{aligned}$$

It follows that

$$[\forall x(\varphi \to \psi(x))]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{A}} \Rightarrow [\forall x \psi(x)]_{\mathfrak{A}}.$$

2.

$$\begin{aligned}[\exists x(\varphi \to \psi(x))]_{\mathfrak{A}} &= \bigvee_{a\in\mathfrak{A}} [\varphi \to \psi(a)]_{\mathfrak{A}} = \bigvee_{a\in\mathfrak{A}} [\neg\varphi]_{\mathfrak{A}} \oplus [\psi(a)]_{\mathfrak{A}} \\ &= [\neg\varphi]_{\mathfrak{A}} \oplus \bigvee_{a\in\mathfrak{A}} [\psi(a)]_{\mathfrak{A}} = [\neg\varphi]_{\mathfrak{A}} \oplus [\exists x \psi(x)]_{\mathfrak{A}} \\ &= [\varphi \to \exists x \psi(x)]_{\mathfrak{A}}.\end{aligned}$$

■

We will give some other minor properties of infinite forcing in the appendix, at the end of the article.

5. Generic Structures

As we have seen in the previous section infinite forcing and standard semantics do not coincide, nevertheless the structures in which they do coincide enjoy remarkable properties.

Recall that we work on a fixed Ł∀-language, $\mathcal{L}$ and a fixed inductive class of structures Σ. The following result characterizes the members $\mathfrak{A}$ of Σ for which $[\ ]_{\mathfrak{A}}$ and $\|\ \|_{\mathfrak{A}}$ coincide.

PROPOSITION 5.1. *For any $\mathfrak{A} \in \Sigma$ the following assertions are equivalent:*

(1) $\|\varphi\|_{\mathfrak{A}} = [\varphi]_{\mathfrak{A}}$*, for all sentences* $\mathcal{L}(\mathfrak{A})$*;*

(2) $\|\varphi\|_{\mathfrak{A}} = [\neg\neg\varphi]_{\mathfrak{A}}$*, for all sentences* $\mathcal{L}(\mathfrak{A})$*;*

(3) $[\varphi]_{\mathfrak{A}} \oplus [\neg\varphi]_{\mathfrak{A}} = 1$*, for all sentences* $\mathcal{L}(\mathfrak{A})$*;*

(4) $[\neg\varphi]_{\mathfrak{A}} = [\varphi]^*_{\mathfrak{A}}$*, for all sentences* $\mathcal{L}(\mathfrak{A})$*;*

PROOF. (1) $\Longrightarrow$ (2) By (1) $[\neg\neg\varphi]_{\mathfrak{A}} = \|\neg\neg\varphi\|_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{A}}$

(2) $\Longrightarrow$ (3) We will prove, by induction on the length of the formula, that $[\varphi]_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{A}}$, from this (3) follows trivially. If φ is atomic, then the claim holds by definition. Suppose $\varphi = \neg\psi$, note that from (2) and Lemma 4.7 (item 3), we have $\|\neg\varphi\|_{\mathfrak{A}} = [\neg\neg\neg\varphi]_{\mathfrak{A}} = [\neg\varphi]_{\mathfrak{A}}$. If $\varphi = \psi \to \xi$, then by Lemma 4.6

$$\begin{aligned}[\varphi]_{\mathfrak{A}} = [\psi \to \xi]_{\mathfrak{A}} &= [\neg\psi]_{\mathfrak{A}} \oplus [\xi]_{\mathfrak{A}} = \\ &= \|\neg\psi\|_{\mathfrak{A}} \oplus \|\xi\|_{\mathfrak{A}} = \|\neg\psi \underline{\oplus} \xi\|_{\mathfrak{A}} = \\ &= \|\varphi\|_{\mathfrak{A}}.\end{aligned}$$

Finally, if $\varphi = \exists x\psi(x)$ then the claim follows by definition and the induction hypothesis.

(3) $\Longrightarrow$ (4) According to Lemma 4.7 (item 1), $[\varphi] \odot [\neg\varphi] = 0$, therefore, by using $[\varphi]_{\mathfrak{A}} \oplus [\neg\varphi]_{\mathfrak{A}} = 1$, we obtain $[\neg\varphi]_{\mathfrak{A}} = [\varphi]^*_{\mathfrak{A}}$.

(4) $\Longrightarrow$ (1) By induction on the complexity of φ. We shall treat only the following two cases:

(i) $\varphi = \neg\psi$. By induction hypothesis, $\|\psi\|_{\mathfrak{A}} = [\psi]_{\mathfrak{A}}$, therefore $\|\varphi\|_{\mathfrak{A}} = \|\neg\psi\|_{\mathfrak{A}} = \|\psi\|^*_{\mathfrak{A}} = [\psi]^*_{\mathfrak{A}} = [\neg\psi]_{\mathfrak{A}} = [\varphi]_{\mathfrak{A}}$.

(ii) $\varphi = \psi \to \chi$. By induction hypothesis, $\|\psi\|_{\mathfrak{A}} = [\psi]_{\mathfrak{A}}$ and $\|\chi\|_{\mathfrak{A}} = [\chi]_{\mathfrak{A}}$, therefore, using Lemma 4.5, 2 and (i) above, we get $[\varphi]_{\mathfrak{A}} = [\psi \to \chi]_{\mathfrak{A}} = [\neg\psi]_{\mathfrak{A}} \oplus [\chi]_{\mathfrak{A}} = \|\neg\psi\|_{\mathfrak{A}} \oplus \|\chi\|_{\mathfrak{A}} = \|\neg\psi\underline{\oplus}\chi\|_{\mathfrak{A}} = \|\psi \to \chi\|_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{A}}$.

■

Definition 5.2. A structure $\mathfrak{A} \in \Sigma$ which satisfies the equivalent conditions of Proposition 5.1 will be called Σ-*generic*.

Theorem 5.3. *Any structure $\mathfrak{A} \in \Sigma$ is a substructure of a Σ-generic structure.*

Proof. Let λ be a cardinal such that $\max(card(\mathcal{L}), card(A), 2^{\omega}) < \lambda$. Consider an enumeration $\{\varphi_\alpha : \alpha < \lambda\}$ of the sentences of $\mathcal{L}(\mathfrak{A})$. We shall construct by transfinite induction on the ordinals α a chain of structures $\langle \mathfrak{A}_\alpha : \alpha < \lambda \rangle$ such that the following condition holds:

$$\left[\varphi_\alpha\right]_{\mathfrak{A}_{\alpha+1}} \oplus \left[\neg\varphi_\alpha\right]_{\mathfrak{A}_{\alpha+1}} = 1,$$

for all $\alpha < \lambda$. We must consider the following cases:

- $\alpha = 0$. We take $\mathfrak{A}_0 = \mathfrak{A}$.
- $\alpha = \beta + 1$ is a successor ordinal. The induction hypothesis asserts that the chain $\langle \mathfrak{A}_\xi : \xi \leq \beta \rangle$ there was constructed such that $\left[\varphi_\xi\right]_{\mathfrak{A}_{\xi+1}} \oplus \left[\neg\varphi_\xi\right]_{\mathfrak{A}_{\xi+1}} = 1$ for all $\xi < \beta$. By absurdum, let us assume that

$$[\varphi_\beta]_{\mathfrak{B}} \oplus [\neg\varphi_\beta]_{\mathfrak{B}} < 1 \text{ for all } \mathfrak{B} \in \Sigma \text{ such that } \mathfrak{A}_\beta \subseteq \mathfrak{B}. \qquad \text{(b)}$$

Let us consider a cardinal $k > 2^{\omega}$. We shall define, by transfinite induction on the ordinals ξ, a chain of structures $\langle \mathfrak{A}_{\beta_\xi} : \xi < k \rangle$ such that the following condition holds:

$$\xi < \eta < k \Longrightarrow [\varphi_\beta]_{\mathfrak{A}_{\beta_\xi}} < [\varphi_\beta]_{\mathfrak{A}_{\beta_\eta}}. \qquad \text{(c)}$$

We must take in account the following cases:

- $\xi = 0$. We set $\mathfrak{A}_{\beta_0} = \mathfrak{A}$.
- $\xi = \eta + 1$ is a successor ordinal. By (b) we have $[\varphi_\beta]_{\mathfrak{A}_{\beta_\eta}} \oplus [\neg\varphi_\beta]_{\mathfrak{A}_{\beta_\eta}} < 1$, hence

$$\bigwedge_{\mathfrak{A}_{\beta\eta} \subseteq \mathfrak{C}} ([\varphi_\beta]_{\mathfrak{A}_{\beta_\eta}} \oplus \left[\varphi_\beta\right]^*_{\mathfrak{C}}) = [\varphi_\beta]_{\mathfrak{A}_{\beta_\eta}} \oplus \bigwedge_{\mathfrak{A}_{\beta\eta} \subseteq \mathfrak{C}} \left[\varphi_\beta\right]^*_{\mathfrak{C}} < 1.$$

Thus there exists an extension $\mathfrak{C}$ of $\mathfrak{A}_{\beta\eta}$ such that

$$[\varphi\beta]_{\mathfrak{C}} \Rightarrow [\varphi\beta]_{\mathfrak{A}_{\beta\eta}} = [\varphi\beta]^*_{\mathfrak{C}} \oplus [\varphi\beta]_{\mathfrak{A}_{\beta\eta}} < 1,$$

hence $[\varphi\beta]_{\mathfrak{C}} \not\leq [\varphi\beta]_{\mathfrak{A}_{\beta\eta}}$, i.e. $[\varphi\beta]_{\mathfrak{A}_{\beta\eta}} < [\varphi\beta]_{\mathfrak{C}}$.
Then we define $\mathfrak{A}_{\beta_\xi} = \mathfrak{C}$.

– ξ is a limit ordinal. Set

$$\mathfrak{A}_{\beta_\xi} = \bigcup_{\eta<\xi} \mathfrak{A}_{\beta\eta}.$$

We must prove that $[\varphi\beta]_{\mathfrak{A}_{\beta\eta}} < [\varphi\beta]_{\mathfrak{A}_{\beta_\xi}}$ for all ordinals $\eta < \xi$. By absurdum, we suppose that there exists $\eta < \xi$ such that $[\varphi\beta]_{\mathfrak{A}_{\beta\eta}} = [\varphi\beta]_{\mathfrak{A}_{\beta_\xi}}$. If we consider the ordinal $\nu = \eta + 1$ then

$$[\varphi\beta]_{\mathfrak{A}_{\beta\eta}} < [\varphi\beta]_{\mathfrak{A}_{\beta\nu}} \leq [\varphi\beta]_{\mathfrak{A}_{\beta_\xi}} = [\varphi\beta]_{\mathfrak{A}_{\beta\eta}}.$$

We have obtained a contradiction, therefore $[\varphi\beta]_{\mathfrak{A}_{\beta\eta}} < [\varphi\beta]_{\mathfrak{A}_{\beta_\xi}}$. In this way, the construction of the chain $\langle \mathfrak{A}_{\beta_\xi} : \xi < k \rangle$ is finished.
From (c) one infers that $2^\omega \geq k$. This contradiction shows that (b) fails, i.e. $[\varphi\beta]_{\mathfrak{C}} \oplus [\neg\varphi\beta]_{\mathfrak{C}} = 1$ for some extension $\mathfrak{C}$ of $\mathfrak{A}_\beta$. Then we set $\mathfrak{A}_\alpha = \mathfrak{C}$.

- α is a limit ordinal. We define

$$\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta.$$

Let us denote

$$\mathfrak{A}^{(1)} = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha.$$

Then we get $[\varphi]_{\mathfrak{A}^{(1)}} \oplus [\neg\varphi]_{\mathfrak{A}^{(1)}} = 1$ for all sentences φ of $\mathcal{L}(\mathfrak{A})$. Using this procedure we obtain a chain of structures $\mathfrak{A} \subseteq \mathfrak{A}^{(1)} \subseteq ... \subseteq \mathfrak{A}^{(n)} \subseteq ...$ such that $[\varphi]_{\mathfrak{A}^{(n+1)}} \oplus [\neg\varphi]_{\mathfrak{A}^{(n+1)}} = 1$ for all sentences φ of $\mathcal{L}(\mathfrak{A}^{(n)})$. Let us denote

$$\mathfrak{A}_\omega = \bigcup_{n<\omega} \mathfrak{A}^{(n)}.$$

If φ is a sentence of $\mathcal{L}(\mathfrak{A}_\omega)$ then there exists $n \in \omega$ such that φ is in the language $\mathcal{L}(\mathfrak{A}_n)$, hence

$$[\varphi]_{\mathfrak{A}_\omega} \oplus [\neg\varphi]_{\mathfrak{A}_\omega} \geq [\varphi]_{\mathfrak{A}^{(n+1)}} \oplus [\neg\varphi]_{\mathfrak{A}^{(n+1)}} = 1.$$

Thus $\mathfrak{A}_\omega$ is a Σ-generic structure and $\mathfrak{A} \subseteq \mathfrak{A}_\omega$.

■

We will now characterize the class of Σ-generic structures. The following definition will turn out to be relevant to our interests.

DEFINITION 5.4. A structure $\mathfrak{A} \in \Sigma$ is said to be Σ-*existentially complete* if for any extension $\mathfrak{B} \in \Sigma$ of $\mathfrak{A}$ and for any existential sentence φ of $\mathcal{L}(\mathfrak{A})$, we have $\|\varphi\|_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{B}}$.

LEMMA 5.5. *If* $\mathfrak{A}$, $\mathfrak{B}$ *are two* Σ-*generic structures such that* $\mathfrak{A} \subseteq \mathfrak{B}$ *then* $\mathfrak{A} \preceq \mathfrak{B}$.

PROOF. According to Lemma 4.6, we have for any sentence φ of $\mathcal{L}(\mathfrak{A})$:

$$\|\varphi\|_{\mathfrak{A}} = [\varphi]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{B}} = \|\varphi\|_{\mathfrak{B}},$$
$$\|\varphi\|^*_{\mathfrak{A}} = \|\neg\varphi\|_{\mathfrak{A}} = [\neg\varphi]_{\mathfrak{A}} \leq [\neg\varphi]_{\mathfrak{B}} = \|\neg\varphi\|_{\mathfrak{B}} = \|\varphi\|^*_{\mathfrak{B}},$$

therefore $\|\varphi\|_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{B}}$. ■

In the previous section we noticed that although not completely, infinite forcing value and truth value are similar. Next lemma formalizes this similarity until the level of existential formulas.

LEMMA 5.6. *Let* $\mathfrak{A} \in \Sigma$. *For any existential sentence* φ *of* $\mathcal{L}(\mathfrak{A})$ *we have* $[\varphi]_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{A}}$.

PROOF. First we shall prove that for any quantifier-free formula $\varphi(x_1, ..., x_n)$ of $\mathcal{L}$ and for any $a_1, ..., a_n \in A$ we have $[\varphi(a_1, ..., a_n)]_{\mathfrak{A}} = \|\varphi(a_1, ..., a_n)\|_{\mathfrak{A}}$. This assertion follows by induction on the complexity of the formula φ.

- If φ is atomic then we apply Definition 4.3, 1.
- $\varphi = \neg\psi(x_1, ..., x_n)$. By induction hypothesis we have $[\psi(a_1, ..., a_n)]_{\mathfrak{B}} = \|\psi(a_1, ..., a_n)\|_{\mathfrak{B}}$ for all $\mathfrak{B} \in \Sigma$. Thus

$$\begin{aligned}[\varphi(a_1, ..., a_n)]_{\mathfrak{A}} &= \bigwedge_{\mathfrak{A} \subseteq \mathfrak{B}} [\psi(a_1, ..., a_n)]^*_{\mathfrak{B}} = \bigwedge_{\mathfrak{A} \subseteq \mathfrak{B}} \|\psi(a_1, ..., a_n)\|^*_{\mathfrak{B}} \\ &= \bigwedge_{\mathfrak{A} \subseteq \mathfrak{B}} \|\neg\psi(a_1, ..., a_n)\|_{\mathfrak{B}} = \|\neg\psi(a_1, ..., a_n)\|_{\mathfrak{A}} \\ &= \|\varphi(a_1, ..., a_n)\|,\end{aligned}$$

where we used the fact that $\mathfrak{A} \subseteq \mathfrak{B}$

- $\varphi = \psi(x_1, ..., x_n) \to \chi(x_1, ..., x_n)$. According to Lemma 4.5, 2 and the previous point we get

$$\begin{aligned}[\varphi(a_1, ..., a_n)]_{\mathfrak{A}} &= [\neg\psi(a_1, ..., a_n)]_{\mathfrak{A}} \oplus [\chi(a_1, ..., a_n)]_{\mathfrak{A}} \\ &= \|\neg\psi(a_1, ..., a_n)\|_{\mathfrak{A}} \oplus \|\chi(a_1, ..., a_n)\|_{\mathfrak{A}} \\ &= \|\varphi(a_1, ..., a_n)\|_{\mathfrak{A}}.\end{aligned}$$

Now we consider an existential sentence $\exists x_1, ..., x_n\varphi(x_1, ..., x_n)$. By applying the previous remark one obtains

$$\begin{aligned}[\exists x_1 ... \exists x_n \varphi(x_1, ..., x_n)]_{\mathfrak{A}} &= \bigvee \left\{[\varphi(a_1, ..., a_n)]_{\mathfrak{A}} \mid a_1, ..., a_n \in \mathfrak{A}\right\} \\ &= \bigvee \left\{\|\varphi(a_1, ..., a_n)\|_{\mathfrak{A}} \mid a_1, ..., a_n \in A\right\} \\ &= \|\exists x_1 ... \exists x_n \varphi(x_1, ..., x_n)\|_{\mathfrak{A}}.\end{aligned}$$

■

Proposition 5.7. *Any Σ-generic structure $\mathfrak{A}$ is Σ-existentially-complete.*

Proof. Consider an extension $\mathfrak{B} \in \Sigma$ of $\mathfrak{A}$ and φ an existential sentence of $\mathcal{L}(\mathfrak{A})$. Since $[\varphi]_{\mathfrak{B}} \odot [\neg\varphi]_{\mathfrak{B}} = 0$ and $[\neg\varphi]_{\mathfrak{A}} \leq [\neg\varphi]_{\mathfrak{B}}$ it follows that $[\varphi]_{\mathfrak{B}} \odot [\neg\varphi]_{\mathfrak{A}} = 0$. $\mathfrak{A}$ is Σ-generic, so $[\neg\varphi]_{\mathfrak{A}} = [\varphi]^*_{\mathfrak{A}}$, hence $[\varphi]_{\mathfrak{B}} \odot [\varphi]^*_{\mathfrak{A}} = 0$, i.e. $[\varphi]_{\mathfrak{B}} \leq [\varphi]_{\mathfrak{A}}$. By Lemma 4.6, we get $[\varphi]_{\mathfrak{A}} = [\varphi]_{\mathfrak{B}}$. Using Lemma 5.6, we obtain $\|\varphi\|_{\mathfrak{A}} = \|\varphi\|_{\mathfrak{B}}$, so $\mathfrak{A}$ is Σ-existentially complete. ■

Corollary 5.8. *Any structure $\mathfrak{A} \in \Sigma$ is a substructure of an Σ-existentially complete structure.*

Proof. By Theorem 5.3 and Proposition 5.7. ■

Let use denote by $\mathfrak{G}_\Sigma$ the class of Σ-generic structures.

Proposition 5.9. *$\mathfrak{G}_\Sigma$ is an inductive class.*

Proof. Let $< \mathfrak{A}_\xi : \xi < \lambda >$ be a chain of Σ-generic structures and

$$\mathfrak{A} = \bigcup_{\xi<\lambda} \mathfrak{A}_\xi.$$

Let us consider a sentence φ of $\mathcal{L}(\mathfrak{A})$. Then there exists an ordinal $\xi < \lambda$ such that φ is a sentence of $\mathcal{L}(\mathfrak{A}_\xi)$. Since $\mathfrak{A}_\xi$ is Σ-generic, we get $[\varphi]_{\mathfrak{A}_\xi} \oplus [\neg\varphi]_{\mathfrak{A}_\xi} = 1$. By Lemma 4.6 it follows that $[\varphi]_{\mathfrak{A}} \oplus [\neg\varphi]_{\mathfrak{A}} = 1$, hence $\mathfrak{A}$ is Σ-generic. ■

THEOREM 5.10. $\mathfrak{G}_\Sigma$ *is the unique subclass of* Σ *satisfying the following properties:*

(1) *it is model-consistent with* Σ*;*

(2) *it is model-complete;*

(3) *it is maximal with respect to* (1) *and* (2)*.*

PROOF. By Lemma 5.5 and Theorem 5.3, $\mathfrak{G}_\Sigma$ satisfies (1) and (2). Assume that Σ' satisfies (1) and (2). We must prove $\Sigma' \subseteq \mathfrak{G}_\Sigma$. Let $\mathfrak{A} \in \Sigma'$, then, by (1) there is an extension $\mathfrak{A}_1$ in Σ, which in turn has an extension $\mathfrak{A}_2$ in $\mathfrak{G}_\Sigma$. Reiterating this process we end up with an infinite chain of structures:

$$\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq ... \subseteq \mathfrak{A}_n \subseteq ...,$$

which can be reduced to a chain of structures belonging either to Σ' or $\mathfrak{G}_\Sigma$:

$$\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_2 \subseteq ... \subseteq \mathfrak{A}_{2n} \subseteq ... .$$

By (2), the subchains $\langle \mathfrak{A}_{4n} \rangle_{n \in \mathbb{N}} \in \Sigma'$ and $\langle \mathfrak{A}_{4n+2} \rangle_{n \in \mathbb{N}} \in \mathfrak{G}_\Sigma$, are both elementary. We set

$$\mathfrak{A}_\omega = \bigcup_{n \in \omega} \mathfrak{A}_{4n} = \bigcup_{n \in \omega} \mathfrak{A}_{4n+2}.$$

By Proposition 5.9, $\mathfrak{A}_\omega \in \mathfrak{G}_\Sigma$, hence by point (2) in Proposition 5.1, for any φ of $\mathcal{L}(\mathfrak{A}_\omega)$, $[\neg\neg\varphi]_{\mathfrak{A}_\omega} = \|\varphi\|_{\mathfrak{A}_\omega}$. But if ψ is a $\mathcal{L}(\mathfrak{A})$ formula, by Lemma 4.6, $[\neg\neg\psi]_{\mathfrak{A}} \leq [\neg\neg\psi]_{\mathfrak{A}_\omega}$ and $\|\psi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}_\omega}$ because $\mathfrak{A} \preceq \mathfrak{A}_\omega$, whence $[\neg\neg\psi]_{\mathfrak{A}} \leq \|\psi\|_{\mathfrak{A}}$.

Let now $\mathfrak{B} \in \Sigma$ be an extension of $\mathfrak{A}$. Let us build a chain of structures using (1) as above:

$$\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{B}' \subseteq \mathfrak{A}_1 \subseteq ... \subseteq \mathfrak{A}_n \subseteq ...,$$

with $\mathfrak{B}' \in \Sigma'$, $\mathfrak{A}_1 \in \Sigma$, $\mathfrak{A}_2 \in \mathfrak{G}_\Sigma$, etc., then if ψ is a $\mathcal{L}(\mathfrak{A})$ formula:

$$[\psi]_{\mathfrak{B}} \leq [\psi]_{\mathfrak{A}_2} = \|\psi\|_{\mathfrak{A}_2} = \|\psi\|_{\mathfrak{A}_\omega} = \|\psi\|_{\mathfrak{A}}.$$

The inequality $[\psi]_{\mathfrak{B}} \leq \|\psi\|_{\mathfrak{A}}$ holds for any extension $\mathfrak{B} \in \Sigma$ of $\mathfrak{A}$, therefore

$$[\neg\psi]^*_{\mathfrak{A}} = (\bigwedge_{\mathfrak{A} \subseteq \mathfrak{B}} [\psi]^*_{\mathfrak{B}})^* = \bigvee_{\mathfrak{A} \subseteq \mathfrak{B}} [\psi]_{\mathfrak{B}} \leq \|\psi\|_{\mathfrak{A}}.$$

Then $\|\psi\|^*_{\mathfrak{A}} \leq [\neg\psi]_{\mathfrak{A}}$. Therefore for any sentence ψ of $\mathcal{L}(\mathfrak{A})$ we have $\|\psi\|_{\mathfrak{A}} = \|\neg\neg\psi\|_{\mathfrak{A}} = \|\neg\psi\|^*_{\mathfrak{A}} \leq [\neg\neg\psi]_{\mathfrak{A}}$, i.e. $\mathfrak{A}$ is Σ-generic (by Proposition 5.1, (2)). ■

Notice that in the classical case a class Λ of structures which is model consistent with σ and model-complete (both concepts here are intended in their classical meaning), then Λ is called the *model companion* of Σ.

In the classical case the model companion is always unique, whereas in our case we were not able to prove such a result, indeed our characterization asks for a maximal class w.r.t. these properties. Whence the necessity of such a maximality requirement in Theorem 5.10 is an open problems.

The interested reader will find other questions which arose during our research as well as some expected applications of the results presented so far in the next section.

6. Final Remarks and Future Works

We have shown that, in the particular case of Łukasiewicz predicate logic, the classical notion of model theoretic forcing can be extended to many-valued logic. Given the absence of standard completeness for the first order versions of those logics, such kind of result can be very useful to study standard structures by model theoretic tools. Of course the plethora of techniques used in model theory have to be, where this is possible, readapted to many-valued logics. The lack of such notions is an obstacle also to understand how far the technique introduced in this article can go and which classical results may have a role also in this framework. Such a study could give new information about a part of logic which is so far mainly unknown.

In classical model theory, Keisler's Generic Model Theorem is an efficient tool for obtaining alternative proofs of some fundamental theorems (completeness, omitting types, interpolation, etc). An open problem is to use Theorem 3.8 to prove similar results in model theory of Łukasiewicz logic or Pavelka logic. Particularly, it would be interesting to study whether the omitting types theorem of [24] can be derived using Theorem 3.8.

In the case of infinite forcing, we proved that any Ł$\forall$-structure can be embedded in a generic one (Theorem 5.3). This is a strong property, from which we derive that any structure can be embedded in an existentially-complete structure, generalizing an important theorem from classical model theory and algebra (see [17]). In the meantime, from Theorem 5.3 it follows a global characterization of the class of generic structures in terms of model-consistency and model-completeness. This again extends a Robinson's theorem of [26].

A new proof of Chang's omitting types theorem [8] was obtained by Lablanquie [20] by Robinson infinite forcing technique. An open problem is to use our infinite forcing machinery to prove a similar result in the framework of Łukasiewicz and Pavelka logics.

Appendix

We give here some other properties of infinite forcing. Such results will help to understand better how the infinite forcing behave in respect to the connectives.

LEMMA 6.1.

1. $[\varphi \underline{\odot} \psi]_{\mathfrak{A}} = \bigwedge_{\mathfrak{A} \subseteq \mathfrak{B}} \left([\neg\varphi]^*_{\mathfrak{B}} \odot [\neg\psi]^*_{\mathfrak{B}}\right)$;
2. $[\varphi \underline{\odot} \psi]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{A}} \odot [\psi]_{\mathfrak{A}}$;
3. $[\varphi \underline{\odot} \varphi]_{\mathfrak{A}} = [\neg\neg\varphi]_{\mathfrak{A}} \odot [\neg\neg\varphi]_{\mathfrak{A}}$;
4. $[\varphi \to \psi]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{A}} \Rightarrow [\psi]_{\mathfrak{A}}$.
5. $[\psi]_{\mathfrak{A}} \leq [\varphi \to \psi]_{\mathfrak{A}}$;
6. $[\varphi \underline{\odot} (\varphi \to \psi)]_{\mathfrak{A}} \leq [\psi]_{\mathfrak{A}}$;
7. $[\varphi \to \psi]_{\mathfrak{A}} \odot [\psi \to \xi]_{\mathfrak{A}} \leq [\varphi \to \xi]_{\mathfrak{A}}$;
8. $[\varphi \to (\psi \to \xi)]_{\mathfrak{A}} = [\psi \to (\varphi \to \xi)]_{\mathfrak{A}}$.

PROOF. 1.

$$\begin{aligned}[\varphi \underline{\odot} \psi]_{\mathfrak{A}} = [\neg(\varphi \to \neg\psi)]_{\mathfrak{A}} &= \bigwedge_{\mathfrak{A} \subseteq \mathfrak{B}} [\varphi \to \neg\psi]^*_{\mathfrak{B}} \\ &= \bigwedge_{\mathfrak{A} \subseteq \mathfrak{B}} \left([\neg\varphi]_{\mathfrak{B}} \oplus [\neg\psi]_{\mathfrak{B}}\right)^* \\ &= \bigwedge_{\mathfrak{A} \subseteq \mathfrak{B}} \left([\neg\varphi]^*_{\mathfrak{B}} \odot [\neg\psi]^*_{\mathfrak{B}}\right).\end{aligned}$$

2. Let $\mathfrak{B}$ be an extension of $\mathfrak{A}$. By Lemmas 4.6 and 4.7 (item 1):

$$[\varphi]_{\mathfrak{A}} \odot [\neg\varphi]_{\mathfrak{B}} \leq [\varphi]_{\mathfrak{B}} \odot [\neg\varphi]_{\mathfrak{B}} = 0;$$

hence $[\varphi]_{\mathfrak{A}} \leq [\neg\varphi]^*_{\mathfrak{B}}$. Similarly, $[\psi]_{\mathfrak{A}} \leq [\neg\psi]^*_{\mathfrak{B}}$, hence

$$[\varphi]_{\mathfrak{A}} \odot [\psi]_{\mathfrak{B}} \leq [\neg\varphi]^*_{\mathfrak{B}} \odot [\neg\psi]^*_{\mathfrak{B}}.$$

This last inequality holds for any extension $\mathfrak{B}$ of $\mathfrak{A}$, therefore, by 1 we obtain the inequality 2.

3. By (1):

$$[\varphi\underline{\odot}\varphi]_{\mathfrak{A}} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} \big([\neg\varphi]^*_{\mathfrak{B}} \odot [\neg\varphi]^*_{\mathfrak{B}}\big)$$
$$= (\bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} [\neg\varphi]^*_{\mathfrak{B}}) \odot (\bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} [\neg\varphi]^*_{\mathfrak{B}})$$
$$= [\neg\neg\varphi]_{\mathfrak{A}} \odot [\neg\neg\varphi]_{\mathfrak{A}}.$$

4.

$$[\varphi]_{\mathfrak{A}} \odot [\varphi \to \psi]_{\mathfrak{A}} = [\varphi]_{\mathfrak{A}} \odot \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} ([\varphi]_B \Rightarrow [\psi]_{\mathfrak{A}})$$
$$\leq [\varphi]_{\mathfrak{A}} \odot ([\varphi]_{\mathfrak{A}} \Rightarrow [\psi]_{\mathfrak{A}})$$
$$\leq [\psi]_{\mathfrak{A}}.$$

Thus the desired inequality follows.

5. For any $\mathfrak{B} \in \Sigma$, $[\psi]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{B}} \Rightarrow [\psi]_{\mathfrak{A}}$, hence

$$[\psi]_{\mathfrak{A}} \leq \bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} ([\varphi]_{\mathfrak{B}} \Rightarrow [\psi]_{\mathfrak{A}}) = [\varphi \to \psi]_{\mathfrak{A}}.$$

6. According to point 2 and 5 we get

$$[\varphi \odot (\varphi \to \psi)]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{A}} \odot [\varphi \to \psi]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{A}} \odot ([\varphi]_{\mathfrak{A}} \Rightarrow [\psi]_{\mathfrak{A}}) \leq [\psi]_{\mathfrak{A}}.$$

7. By Lemma 4.6 $[\varphi]_{\mathfrak{B}} \Rightarrow [\psi]_{\mathfrak{A}} \leq [\varphi]_{\mathfrak{B}} \Rightarrow [\psi]_{\mathfrak{B}}$. Then

$$[\varphi \to \psi]_{\mathfrak{A}} \odot [\psi \to \xi]_{\mathfrak{A}} = \left(\bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} ([\varphi]_{\mathfrak{B}} \Rightarrow [\psi]_{\mathfrak{A}})\right) \odot \left(\bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} ([\psi]_{\mathfrak{B}} \Rightarrow [\xi]_{\mathfrak{A}}\right)$$
$$\leq \left(\bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} ([\varphi]_{\mathfrak{B}} \Rightarrow [\psi]_{\mathfrak{B}})\right) \odot \left(\bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} ([\psi]_{\mathfrak{B}} \Rightarrow [\xi]_{\mathfrak{A}})\right)$$
$$\leq \bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} \big(([\varphi]_{\mathfrak{B}} \Rightarrow [\psi]_{\mathfrak{B}}) \odot ([\psi]_{\mathfrak{B}} \Rightarrow [\xi]_{\mathfrak{A}})\big)$$
$$\leq \bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} ([\varphi]_{\mathfrak{B}} \Rightarrow [\xi]_{\mathfrak{A}})$$
$$= [\varphi \to \xi]_{\mathfrak{A}}.$$

8.

$$\begin{aligned}
[\varphi \to (\psi \to \xi)]_{\mathfrak{A}} &= \bigwedge_{\mathfrak{B} \supseteq \mathfrak{A}} ([\varphi]_{\mathfrak{B}} \Rightarrow [\psi \to \xi]_{\mathfrak{A}}) \\
&= \bigwedge_{\mathfrak{B} \supseteq \mathfrak{A}} \left([\varphi]_{\mathfrak{B}} \Rightarrow \bigwedge_{\mathfrak{C} \supseteq \mathfrak{A}} ([\psi]_{\mathfrak{C}} \Rightarrow [\xi]_{\mathfrak{A}}) \right) \\
&= \bigwedge_{\mathfrak{B}, \mathfrak{C} \supseteq \mathfrak{A}} \big([\varphi]_{\mathfrak{B}} \Rightarrow ([\psi]_{\mathfrak{C}} \Rightarrow [\xi]_{\mathfrak{A}}) \big) \\
&= \bigwedge_{\mathfrak{B}, \mathfrak{C} \supseteq \mathfrak{A}} ([\varphi]_{\mathfrak{B}} \odot [\psi]_{\mathfrak{C}} \Rightarrow [\xi]_{\mathfrak{A}}) \,.
\end{aligned}$$

A similar computation shows that $[\psi \to (\varphi \to \xi)]_{\mathfrak{A}}$ has the same value.

■

We say that a class Σ has the amalgamation property (AP) if for any two extensions $\mathfrak{B}$ and $\mathfrak{C}$ of $\mathfrak{A}$ there exists a common extension $\mathfrak{D}$ of $\mathfrak{B}$ and $\mathfrak{C}$.

PROPOSITION 6.2. *Assume that Σ satisfies AP. Then for any formulas $\varphi(x)$ and $\psi(x)$ of $Ł\forall$, the following inequalities hold:*

1. $[\forall x(\varphi(x) \to \psi(x))]_{\mathfrak{A}} \leq [\forall x \varphi(x)]_{\mathfrak{A}} \Rightarrow [\forall x \psi(x)]_{\mathfrak{A}}$;
2. $[\forall x(\varphi(x) \to \psi(x))]_{\mathfrak{A}} \odot [\forall x(\psi(x) \to \xi(x))]_{\mathfrak{A}} \leq [\forall x(\varphi(x) \Rightarrow \xi(x))]_{\mathfrak{A}}$.

PROOF. 1. Let $\mathfrak{B}$ an extension of $\mathfrak{A}$ and $b \in \mathfrak{B}$. Assume $\mathfrak{C}$ and $\mathfrak{D}$ two arbitrary extensions of $\mathfrak{B}$, hence, by AP, there is a common extension of $\mathfrak{C}$ and $\mathfrak{D}$, let us call it $\mathfrak{F}$. Then, by Lemmas 4.6 and 6.1

$$\begin{aligned}
&[\varphi(b)]_{\mathfrak{C}} \odot [\varphi(b) \to \psi(b)]_{\mathfrak{D}} \leq [\varphi(b)]_{\mathfrak{F}} \odot [\varphi(b) \to \psi(b)]_{\mathfrak{F}} \leq \\
&\leq [\psi(b)]_{\mathfrak{F}} \leq \bigvee_{\mathfrak{H} \supseteq \mathfrak{B}} [\psi(b)]_{\mathfrak{H}}.
\end{aligned}$$

These inequalities hold for any extension $\mathfrak{C}$ or $\mathfrak{D}$ of $\mathfrak{B}$, hence

$$\begin{aligned}
[\forall x(\varphi(x) \to \psi(x))]_{\mathfrak{A}} &\odot [\forall x \varphi(x)]_{\mathfrak{A}} \leq \\
&\leq \left(\bigvee_{\mathfrak{C} \supseteq \mathfrak{B}} [\varphi(b)]_{\mathfrak{C}} \right) \odot \left(\bigvee_{\mathfrak{D} \supseteq \mathfrak{B}} [\varphi(b) \to \psi(b)]_{\mathfrak{D}} \right) = \\
&= \bigvee_{\mathfrak{C}, \mathfrak{D} \supseteq \mathfrak{B}} ([\varphi(b)]_{\mathfrak{C}} \odot [\varphi(b) \to \psi(b)]_{\mathfrak{D}}) \leq \\
&\leq \bigvee_{\mathfrak{H} \supseteq \mathfrak{B}} [\psi(b)]_{\mathfrak{H}}.
\end{aligned}$$

Then we get

$$[\forall x(\varphi(x) \to \psi(x))]_{\mathfrak{A}} \odot [\forall x\varphi(x)]_{\mathfrak{A}} \leq \bigwedge_{\mathfrak{B}\supseteq\mathfrak{A}} \bigwedge_{b\in\mathfrak{B}} \bigvee_{\mathfrak{H}\supseteq\mathfrak{B}} [\psi(b)]_{\mathfrak{H}} = [\forall x\psi(x)]_{\mathfrak{A}}.$$

Thus the desired inequality holds.

2. Let $\mathfrak{B}$ be an extension of $\mathfrak{A}$ in Σ and $b \in \mathfrak{B}$. Consider two extensions $\mathfrak{C}$ and $\mathfrak{D}$ of $\mathfrak{B}$ in Σ, hence, by AP, there exists a common extension $\mathfrak{F}$ of $\mathfrak{C}$ and $\mathfrak{D}$ in Σ. Therefore, by Lemma 4.6 we have:

$$\begin{aligned}[\varphi(b) \to \psi(b)]_{\mathfrak{C}} \odot [\psi(b) \to \xi(b)]_{\mathfrak{D}} &\leq [\varphi(b) \to \psi(b)]_{\mathfrak{F}} \odot [\psi(b) \to \xi(b)]_{\mathfrak{F}} \\ &\leq [\varphi(b) \to \xi(b)]_{\mathfrak{F}} \\ &\leq \bigvee \left([\varphi(b) \to \xi(b)]_{\mathfrak{H}} \mid \mathfrak{H} \supseteq \mathfrak{B}\right).\end{aligned}$$

It follows that for any extension $\mathfrak{B}$ of $\mathfrak{A}$ and $b \in \mathfrak{B}$ we get

$$\begin{aligned}&[\forall x(\varphi(x) \to \psi(x))]_{\mathfrak{A}} \odot [\forall x(\psi(x) \to \xi(x))]_{\mathfrak{A}} \\ &\leq \bigvee_{\mathfrak{C}\supseteq\mathfrak{B}} ([\varphi(b) \to \psi(b)]_{\mathfrak{C}}) \odot \bigvee_{\mathfrak{D}\supseteq\mathfrak{B}} ([\psi(b) \to \xi(b)]_{\mathfrak{D}}) \\ &= \bigvee_{\mathfrak{C},\mathfrak{D}\supseteq\mathfrak{B}} ([\varphi(b) \to \psi(b)]_{\mathfrak{C}} \odot [\psi(b) \to \xi(b)]_{\mathfrak{D}}) \\ &\leq \bigwedge_{\mathfrak{B}} \bigwedge_{b\in\mathfrak{B}} \bigvee_{\mathfrak{H}\supseteq\mathfrak{B}} ([\varphi(b) \to \xi(b)]_{\mathfrak{H}}) \\ &= [\forall x(\varphi(x) \to \xi(x))]_{\mathfrak{A}}.\end{aligned}$$

■

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