A short introduction to formal fuzzy logic via t-norms

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1 Introduction

Since it was conceived in 1965 at the Berkeley University, fuzzy logic raised several comments, both skeptical and enthusiastic; nevertheless, after more than forty years, its applications become more and more spread and advanced. They range, nowadays, from agriculture to artificial intelligence, going through telecommunications, economics, medicine, etc. The reasons of such a success are evident: fuzzy logic (and, more generally, fuzzy sets theory) enables to reason about facts whose descriptions are *inherently approximate* and this is an ubiquitous constraint in real life applications.

Fuzzy logic was introduced by Lotfi Zadeh in order to deal with vague predicates; the main idea is to generalize the possible truth values $\{0, 1\}$, to every number in the interval $[0, 1]$. Note that considering only number between 0 and 1 as degree of truth is not restrictive at all, since they form a kind of *generic* setting for truth values, in a sense that will be made precise further on.

Degrees of truth are often confused with probabilities. However, they are conceptually distinct; fuzzy truth represents membership in vaguely defined sets, not likelihood of some event or condition. Moreover probability is not *truth functional*, i.e. the value of a compound event does not depend only on the values of the singular atomic events which compose it.

Albeit with important applications, the approach to inference rules in fuzzy logic can be considered rather näive by a pure logician. For this reason, the effort of a number of logicians, in recent years, has aimed to bear to the discipline foundational basis similar, in rigor, to the one of classical logics (see: [2, 3]). This has been done by considering and further developing a branch of mathematical logic known as *many-valued logic*,

2 Continuous t-norms and residua

The link between many-valued logic and fuzzy logic is given by the concept of **t-norm** [4]. A t-norm is a function $*$ form $[0,1]^2$ to $[0,1]$ which behaves exactly as classical conjunction on the values $\{0,1\}$ and is commutative, associative and non decreasing in both components.
In symbols this amounts to say that for any $x,y,z \in [0,1]$ the following conditions hold:

$$x*0=0 \qquad x*1=x \qquad x*y=y*x$$
$$(x*y)*z=x*(y*z) \qquad \text{if } x \leq y \text{ then } x*z \leq y*z$$

In fuzzy logic, continuous t-norms are often found playing the role of conjunctive connectives.

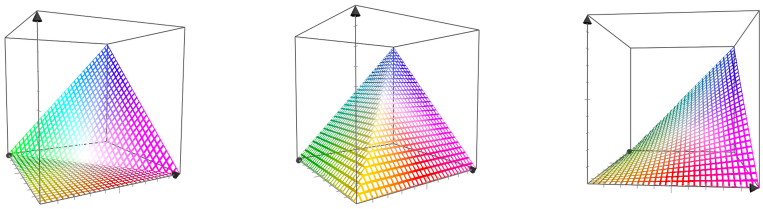
Example 1. The ordinary product in the interval $[0,1]$ of real numbers is a continuous t-norm, called **product conjunction**.
The min function, which gives the minimum between two elements, is a t-norm, called **Gödel conjunction**.
The following function: $x *_L y = \max\{x+y-1,0\}$, which is known as **Łukasiewicz conjunction**, is again a t-norm.

The key result, which ties-up the two systems, is the following theorem [5]

Theorem 2.1. *Every continuous t-norm is locally isomorphic to Łukasiewicz, Gödel or product t-norm.*

So it makes sense to undertake a logical investigation of these three systems in order to give a complete explanation of the properties of fuzzy logic.

Figure 1: Łukasiewicz, Gödel and Product t-norms



It is important to note that in classical logic there is a strict relation between conjunction and implication, which can be exemplified by the equality $(\varphi \& \psi) \to \xi = \varphi \to (\psi \to \xi)$. Such a relation is pervasive in mathematical logic and its applications, and has many interesting interpretation depending on the perspective under which formulas are seen. In general mathematics a similar interdependence exists among two functions, and it is called *residuation*. This is the reason why, in order to have a elegant and well behaved logical system, it is important to interpret implication in the **residuum** of the t-norm.

Definition 2.2. A function $\Rightarrow$ is said to be the residuum of a t-norm $*$ if the following relation holds

$$x * y \leq z \text{ if, and only if, } x \leq y \Rightarrow z$$

Example 2. The function defined as $x \Rightarrow_L y = \min\{1 - x + y, 1\}$ is the residuum of Łukasiewicz conjunction.

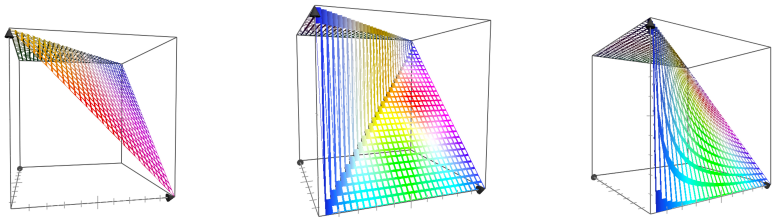
The function $x \Rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$ is the residuum of Gödel conjunction.

The function $x \Rightarrow_\Pi y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$ is the residuum of product conjunction.

Once a t-norm has been fixed, a whole set of connectives can be derived. Indeed it is possible to define a **negation** as $\sim x \stackrel{\text{def}}{\equiv} x \Rightarrow 0$ or a **co-norm**, which plays the role of a disjunction, as $x \oplus y \stackrel{\text{def}}{\equiv} \sim (\sim x \& \sim y)$.

The usual link between syntax and semantic is given by the concept of **evaluations**. Starting form an arbitrary map which sends propositional

Figure 2: Łukasiewicz, Gödel and Product residua



variables to the set $[0,1]$ it is possible to extend it to a function on the whole set of formulas.

Definition 2.3. Given a map form the set of propositional variables to $[0,1]$ we say that $e : \mathit{Formulas} \longrightarrow [0,1]$ is an **evaluation** if :

$$e(\varphi \& \psi) = e(\varphi) * e(\psi)$$

$$e(\varphi \to \psi) = e(\varphi) \Rightarrow e(\psi)$$

$$e(\top) = 1$$

where $\top$ is a symbol for any tautology and $*$ and $\Rightarrow$ are respectively the t-norm and the residuum under consideration.

Note that the concept of evaluation is present also in classical logic and only differs from the one introduced here, by the set of truth values, being $\{0,1\}$ instead of $[0,1]$.

It is important to pause a moment to realize how this approach is close to the one of classical logic.

In classical logic one defines connectives giving their *truth tables*, this is possible since the number of allowed values is finite. However in this generalization, an infinite number of values is considered, thus a complete description of the function often is an analytic form characterizing the semantic of the connective.

In this setting classical results, such as completeness of the logic, have to be found relatively the t-norm. In classical logic one presents an axiom system which describes the behavior of the connectives of classical logic and then one proves that such a system is *complete* w.r.t the truth

tables of the connectives: every formula which is classically true can be derived form the axiom system. This method generalizes to many-valued logic in the following way: once a t-norm (but also a number of t-norms) is chosen one looks for an axiomatic system which is complete w.r.t. the t-norm, i.e. every formula whose evaluation is 1 can be proved from the axioms of the system. This kind of completeness is called **standard completeness**. In more rigorous terms we will say that a logic is standard complete with regard to a (class of) t-norm if, whenever a formula is evaluated to 1 w.r.t. the (class of) t-norm, then it is provable from the axioms of that logic, and vice-versa.

Example 3. Consider the following axioms system:

Definition 2.4. The propositional logic **Ł** is axiomatized as follows:

$$
\begin{array}{ll}
(\varphi \& \psi) \& \xi \leftrightarrow \varphi \& (\phi \& \xi) & \varphi \& \psi \leftrightarrow \psi \& \varphi \\
\neg\neg\varphi \leftrightarrow \varphi & \varphi \& \top = \top \\
(\varphi \to \psi) \to \psi \leftrightarrow (\xi \to \varphi) \to \varphi &
\end{array}
$$

The only logical rule is *modus ponens* (from $\varphi \to \psi$ and φ derive ψ).

It is known [1] that the set of provable formulas of **Ł** are exactly the formulas which, once the t-norm is interpreted as Łukasiewicz t-norm, are sent to 1 by every possible evaluation in oder words the logic **Ł** is standard complete w.r.t. the Łukasiewicz t-norm.

3 Algebraic tools

It is intrinsic to logic to have a some sort of mathematical structure. Once the connectives are understood as functions, the realm they apply to, naturally gets an algebraic structure. In classical propositional logic, the structure arising from this construction are known as boolean algebras. As one may expect, the structures arising from many-valued logics are a generalization of boolean algebra.

One says that a certain class of structures is the **algebraic semantic** of a logic if there is a back and forth translation between formulas of the logic and terms on the structures such that a formula is true if, and only

if, the term is evaluated to 1 in every structure of the class. This can be seen again as a completeness result and it is often called **algebraic completeness**.

Important results of the algebraic theory of many-valued logic state that a certain algebraic semantic is *generated* by some **standard member(s)**. The way a certain class is generated is rather technical, but has the crucial consequence of preserving valid equations. Hence proving that a formula is true in some logic, corresponds to prove that its translation is evaluated to 1 in the standard member of its algebraic semantic.

Note that, in most notable cases the standard members can be just one and very easy to work with. We conclude this section with a number of remarkable examples.

Example 4. Consider the system **Ł** above. The algebraic semantic of **Ł** is known as the class of **MV algebras**. It is known that this class is generated by the standard algebra $\langle [0,1], \odot, \neg, 1 \rangle$ where $\odot$ is Łukasiweciz t-norm and $\neg$ is the negation defined by its residuum. Thus to prove that some formula is true for Łukasiewicz logic, and hence true for every MV algebra, it is sufficient to check whether it is true in the real interval $[0,1]$ endowed with the MV algebra structure given by the Łukasiewicz t-norm.

Example 5. Consider the following system, called **BL** [3].

(i) $(\varphi \& \psi) \to \varphi$

(ii) $(\varphi \to \psi) \to ((\psi \to \theta) \to (\varphi \to \theta))$

(iii) $0 \to \varphi$

(iv) $((\varphi \to \psi) \to \theta) \to (((\psi \to \varphi) \to \theta) \to \theta)$

(v) $(\varphi \& \psi) \to (\psi \& \varphi)$

(vi) $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$

(vii) $(\varphi \to (\psi \to \theta)) \to ((\varphi \& \psi) \to \theta)$

This system is known to be standard complete w.r.t all the continuous t-norm, i.e. a formula is provable form **BL** if, and only if, it is true for every continuous t-norm. In an algebraic perspective this amount to say that a formula is provable from BL if, and only if, it is true in every algebra whose domain is $[0,1]$ and the operation is any continuous t-norm.

4 Fuzzy systems and fuzzy relations

Fuzzy systems are based on rules where it can be recognized the general paradigm "given a condition do an action". In a given formal language, the above paradigm can be coded as a conditional statement: "if $A(x)$ then $B(x)$", where $A(x)$ is the fuzzy conditional part and $B(x)$ is the fuzzy action part of the rule. The most popular model used to codify the link between $A(x)$ and $B(y)$ is a fuzzy relation $R(x,y)$, via the equality

$$B(y) = (A \circ R)(y) = \bigvee_{x \in X} (A(x) \odot R(x,y)).$$

This is called **composition operation** between a fuzzy set $A(x)$ and a fuzzy relation $R(x,y)$. In this setting, the simplicity informally claimed in the literature about the fuzzy rules based description of phenomena, can be seen as the linearity of fuzzy rules. To do this the machinery of semimodules theory in the MV-algebraic setting is introduced. In this context linearity means natural morphism between algebraic structures of same type.

Definition 4.1. A semiring $\mathscr{R} = \langle R, +, \bullet, 0, 1 \rangle$ is an algebraic structure such that: $\langle R, +, 0 \rangle$ is a commutative monoid, $\langle R, \bullet, 1 \rangle$ is a monoid, $\bullet$ distributes over + and $0 \bullet r = r \bullet 0 = 0$, for every $r \in R$. A semiring is called commutative if $\langle R, \bullet, 1 \rangle$is commutative

Example 6. Let A be an MV algebra. Then the reducts $\langle A, \vee, \odot, 0, 1 \rangle$ and$\langle A, \wedge, \oplus, 0, 1 \rangle$ are commutative semirings.

Definition 4.2. Let $\mathscr{R} = \langle R, +, \bullet, 0, 1 \rangle$ be a semiring. A left $\mathscr{R}$-semimodule is a commutative monoid $\langle M, +_M, 0_M \rangle$ endowed with a scalar multiplication $R \times M \to M$, denoted by juxtaposition satisfying, for all $r, s \in R$ and $m, n \in M$, the followings:

$$\begin{aligned} &(r \bullet s)m = r(sm), && && r(m +_M n) = rm +_M rn, \\ &(r + s)m = rm +_M sm, && 1m = m, && r0_M = 0_M = 0m. \end{aligned}$$

Example 7. Let $\mathscr{A} = \langle A, \oplus, \odot, ^*, 0 \rangle 1$ and $\mathscr{B} = \langle B, \oplus, \odot, ^*, 0 \rangle 1$ be MV algebras, $\langle A, \vee, \odot, 0, 1 \rangle$ the semiring reduct of $\mathscr{A}$ and $\langle B, \vee, 0 \rangle$ the sup

monoid reduct of $\mathscr{B}$.
Moreover let h be an MV-homomorphism from $\mathscr{A}$ to $\mathscr{B}$. Then, defining the scalar multiplication, for all $a \in A$ and $b \in B$, by: $ab = h(a) \odot b$, we have $\langle B, \vee, 0 \rangle$ as an example of $\mathscr{A}$-semimodule (MV-semimodule).

Given a MV algebra $\mathscr{A}$ the machinery above allows us to interpret the composition operation as a homomorphism between the functions spaces $\mathscr{A}^X$ and $\mathscr{A}^Y$. Indeed, let $f \in \mathscr{A}^X$ and $R(x,y)$ a fuzzy relation on $X \times Y$, then set

$$H(R)(f)(y) = \bigvee_{x \in X} f(x) \odot R(x,y) = g(y) \in \mathscr{A}^Y.$$

Then we get:

Theorem 4.3. *$H(R)$ is a homomorphism from the $\mathscr{R}$-semimodule $\langle \mathscr{A}^X, \vee, 0 \rangle$ to the R-semimodule $\langle \mathscr{A}^Y, \vee, 0 \rangle$, where $\mathscr{R}$ is the semiring reduct of $\mathscr{A}$.*

The next question is which homomorphisms can be represented by a fuzzy composition. That is: given a homomorphism H from $\mathscr{A}^X$ to $\mathscr{A}^Y$, which conditions on H ensure that there exists a fuzzy relation $R(x,y)$ such that $H = H(R)$. When such a relation exists, we say that H is **representable by a relation**.

For every $a \in X$, denote by f_a the element of A^X, defined as follows:

$$f_a(x) = \begin{cases} 1 & \text{if } x = a\ , \\ 0 & \text{otherwise.} \end{cases}$$

Let $H \in \mathrm{Hom}(A^X, A^Y)$; for every $(x,y) \in X \times Y$, set $R_H(x,y) = (H(f_x))(y)$. With above notations, we get:

Theorem 4.4. *If $H \in \mathrm{Hom}(A^X, A^Y)$ is sup-continuous, then H is representable by a relation as $H(R_H)$.*

5 Conclusions

In this paper we gave some hint on how fuzzy logic can be endowed with formal foundational basis. Although the field is young it has already

given an impressive number of deep results, from (in)completeness results to complexity bounds, characterizations of new logics, automatic theorem provers, etc. The importance of such an approach is twofold. On the one hand, in applications solid grounds are crucial both for a deep understanding of the instruments involved and to know how far they can reach. On the other hand foundational results can serve as a source of inspirations showing new possible (and more effective) ways to the word of applications. Therefore it is advisable a wider perspective in fuzzy logic, which takes in account the importance of solid backgrounds and consider them at any design stage.

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