Two isomorphism criteria

A joint work with V. Marra

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Back&Forth method

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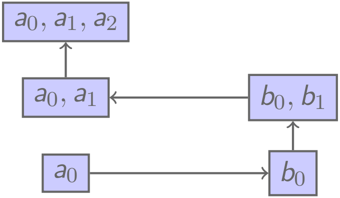

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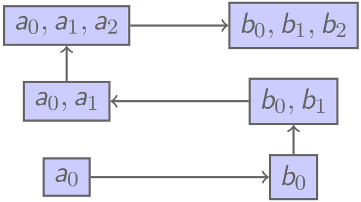

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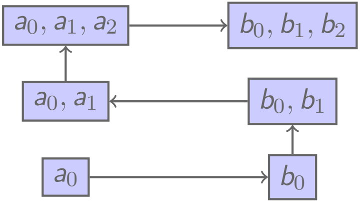

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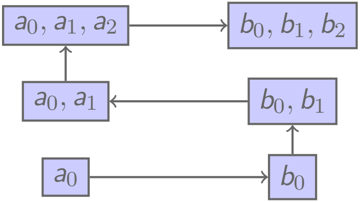

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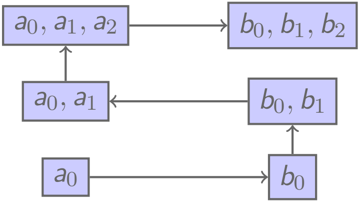

- Any two countably infinite densely ordered sets without endpoints are isomorphic.
- Any two countably infinite atomless Boolean algebras are isomorphic.
- Any two elementary equivalent countable atomic models of a theory are isomorphic.

Ehrenfeucht-Fraïssé games

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Theorem

Suppose the language is finite. Duplicator has a winning strategy iff the two structures are elementarily equivalent

Directed diagrams

- A set I partially ordered by $\leqslant$ is (upward) directed if for any $i, j \in I$ there exists $k \in I$ with $i, j \leqslant k$.

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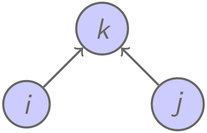

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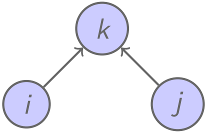

- A directed diagram in C is a pair (B_i, b_{ij}), where
  - I is a directed set, and $i, j \in I$,
  - B_i's are C-object for each $i \in I$,
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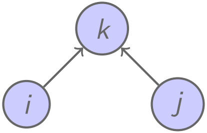

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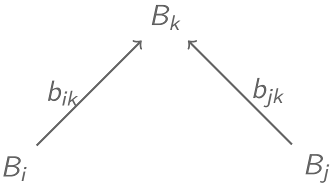

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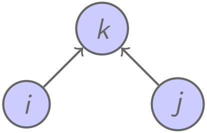

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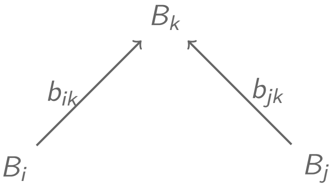
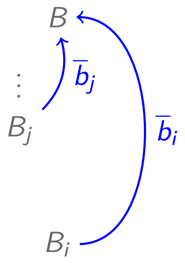


We call the b_{ij}'s the transition morphisms (or maps).

Cocones

A cocone for (B_i, b_{ij}) is a C-object B equipped with C-arrows $\overline{b}_i \colon B_i \to B$ such that



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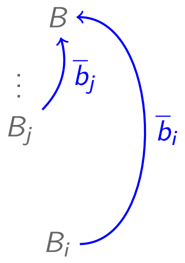

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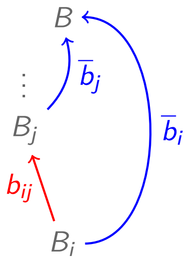

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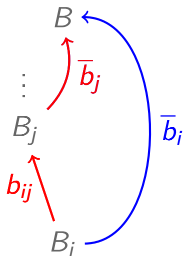
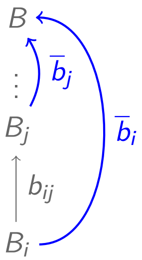


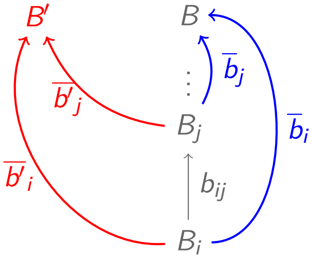
Co-limits

A colimit in C of the diagram (B_i, b_{ij}) is a universal cocone $(B, \overline{b}_i)$,



Co-limits

A colimit in C of the diagram (B_i, b_{ij}) is a universal cocone $(B, \overline{b}_i)$, i.e. for any other cocone $(B',' {}_i)$, there is a unique C-arrow f: $B \to B'$ satisfying $\overline{b}'_i = f \circ \overline{b}_i$ for each $i \in I$.



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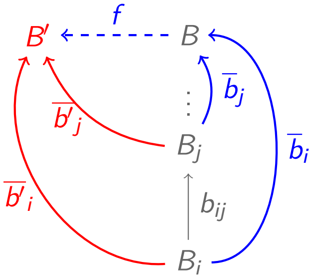

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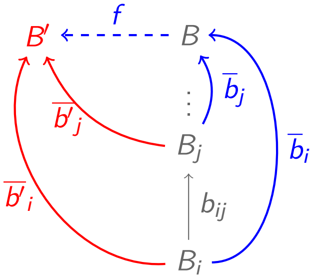


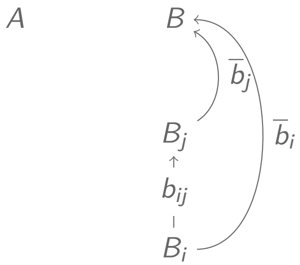
The C-arrows $\overline{b}_i$ are called the colimit morphisms, and B is the colimit object.

Finitely presented object

Definition

An object A in C is finitely presentable if for any arrow f into a colimit C of a diagram $(B, \overline{b}_i)$:

(E)



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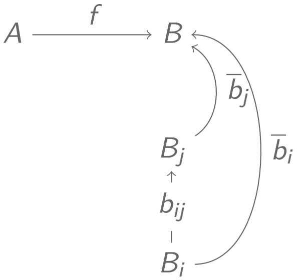

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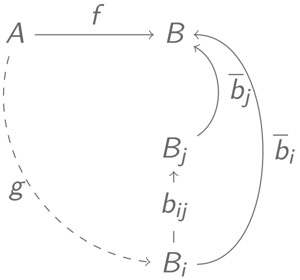

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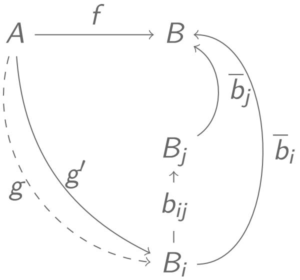
If $g' \colon A \to B_i$ is such that $f = \overline{b}_i \circ g'$, then

(E)



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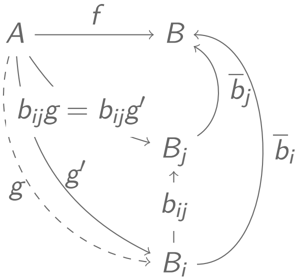
$$\text{If } g'\colon A \to B_i \text{ is such that } f = \overline{b}_i \circ g', \text{ then there is } j \geqslant i \text{ such that } b_{ij} \circ g = b_{ij} \circ g'. \qquad \text{(E)}$$



Finitely generated objects

Remark

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Finitely presented objects are a special case of finitely ***generated*** *objects. They can be characterised as the ones that enjoy properties* (E) *and* (F) *above with regard to colimits in which all arrows are* ***mono****.*

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- Ind-categories allow one to handle "big things in terms of small things".
- Arrows between ind-objects are defined as

$$\text{ind-C}(F, G) := \lim_{d \in D} \operatorname{colim}_{e \in E} \mathrm{C}(Fd, Ge).$$

Confluence

Definition (Confluent sequences)

Two sequences $(A_i, a_i)_{i\in\mathbb{N}}$ and $(B_k, b_k)_{k\in\mathbb{N}}$ are confluent if there exist integers

$$0 < i_1 < i_2 < \cdots \qquad 0 < k_1 < k_2 < \cdots \ ,$$

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$$f_n\colon A_{i_n} \to B_{k_n} \text{ and } g_n\colon B_{k_n} \to A_{i_{n+1}}$$

for each $n \in \mathbb{N}$,

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for each $n \in \mathbb{N}$, such that the following commutativity relations hold:

$$a_{i_n i_{n+1}} = g_n \circ f_n \tag{1}$$

$$b_{k_n k_{n+1}} = f_{n+1} \circ g_n \tag{2}$$

Confluence

$$\cdots \qquad A_{j_1} \xrightarrow{a_{j_1}} A_{j_2} \xrightarrow{a_{j_2}} A_{j_3} \xrightarrow{a_{j_3}} A_{j_4} \xrightarrow{a_{j_4}} A_{j_4} \qquad \cdots$$

$$\cdots \qquad B_{l_1} \xrightarrow[b_{l_1}]{} B_{l_2} \xrightarrow[b_{l_2}]{} B_{l_3} \xrightarrow[b_{l_3}]{} B_{l_4} \xrightarrow[b_{l_4}]{} B_{l_5} \qquad \cdots$$

Confluence

$$
\begin{array}{ccccccccccccc}
\cdots & & A_{j_1} & \xrightarrow{a_{j_1}} & A_{j_2} & \xrightarrow{a_{j_2}} & A_{j_3} & \xrightarrow{a_{j_3}} & A_{j_4} & \xrightarrow{a_{j_4}} & A_{j_4} & & \cdots \\
 & & & \searrow^{f_n} & & & & & & & & & \\
\cdots & & B_{l_1} & \xrightarrow[b_{l_1}]{} & B_{l_2} & \xrightarrow[b_{l_2}]{} & B_{l_3} & \xrightarrow[b_{l_3}]{} & B_{l_4} & \xrightarrow[b_{l_4}]{} & B_{l_5} & & \cdots
\end{array}
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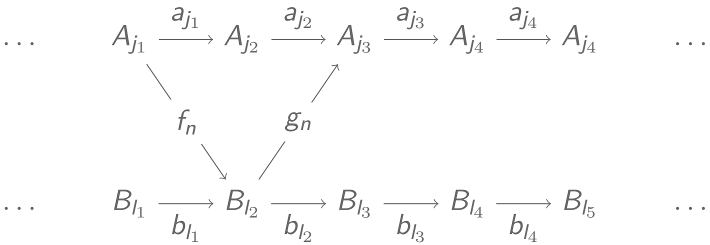
Confluence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f_n
g_n
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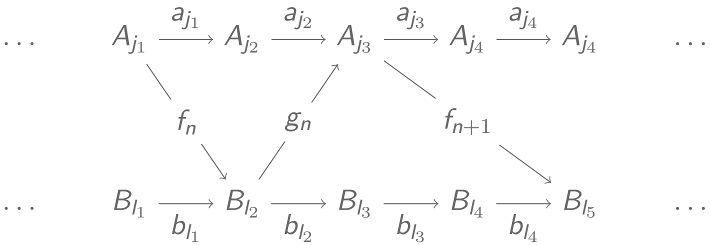
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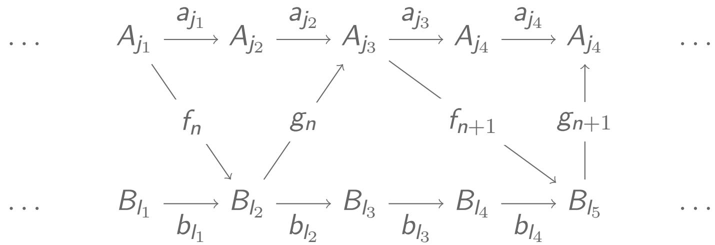
. . .
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A_{j_4}
. . .
f_n
g_n
f_{n+1}
g_{n+1}
. . .
B_{l_1}
b_{l_1}
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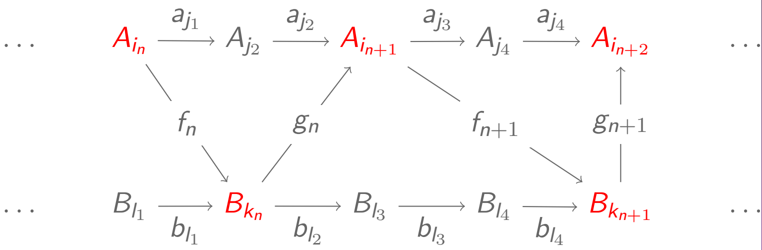
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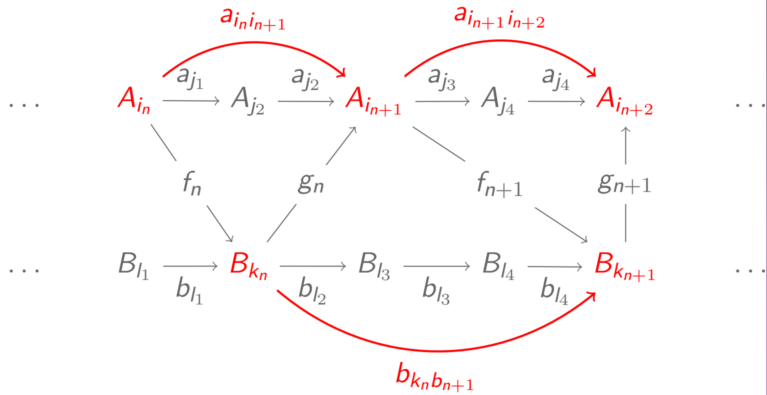

$a_{i_n i_{n+1}}$
$a_{i_{n+1} i_{n+2}}$
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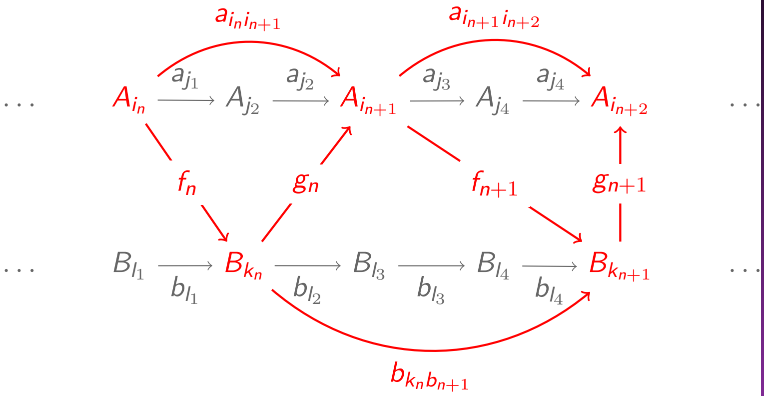

$a_{i_n i_{n+1}}$
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a_{j_1}
a_{j_2}
a_{j_3}
a_{j_4}
$\dots$
A_{i_n}
A_{j_2}
$A_{i_{n+1}}$
A_{j_4}
$A_{i_{n+2}}$
$\dots$
f_n
g_n
f_{n+1}
g_{n+1}
$\dots$
B_{l_1}
B_{k_n}
B_{l_3}
B_{l_4}
$B_{k_{n+1}}$
$\dots$
b_{l_1}
b_{l_2}
b_{l_3}
b_{l_4}
$b_{k_n b_{n+1}}$

The main theorem

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Let C *be any locally small category. Suppose two sequences* $(A_i, a_i)_{i\in\mathbb{N}}$ *and* $(B_k, b_k)_{k\in\mathbb{N}}$ *in* C *admit colimit objects* A *and* B *in* C*, respectively.*

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Then A and B are isomorphic if, and only if, the two sequences are confluent.

A non example

In Set, write $\mathbb{N}$ as a colimit of the sequence

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The two sequences are not confluent. The second sequence does not consist of finitely presentable or finitely generated objects.

Proof

The right-to-left direction (confluence $\Rightarrow$ iso) holds in general i.e. no need for finitely presented objects.

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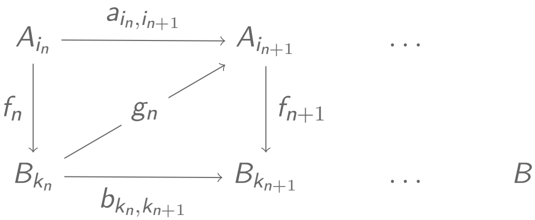
The left-to-right direction (iso $\Rightarrow$ confluence) crucially uses the finite nature of the objects (as seen in the previous example).

Proof of "confluence $\Rightarrow$ iso"



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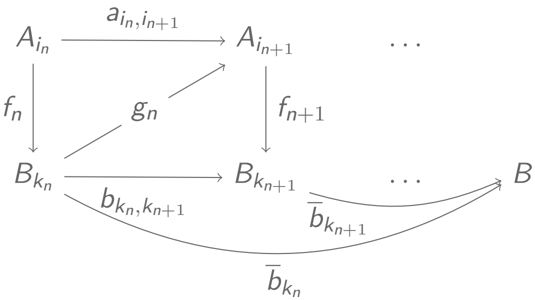
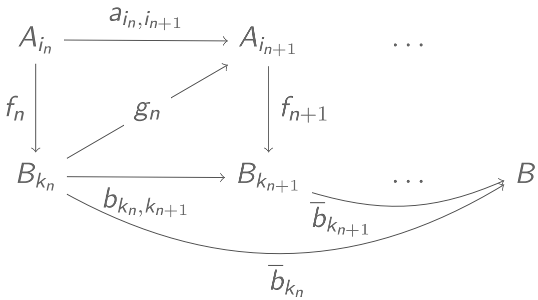


Proof of "confluence ⇒ iso"

Claim

B is a cocone for $(A_i, a_i)_{i\in\mathbb{N}}$



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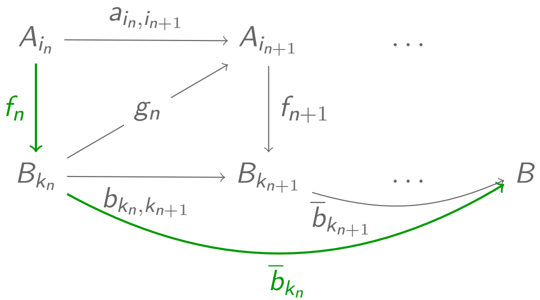

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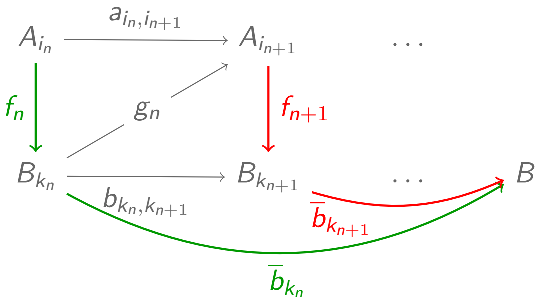

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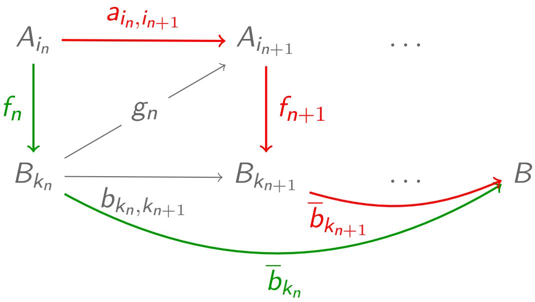

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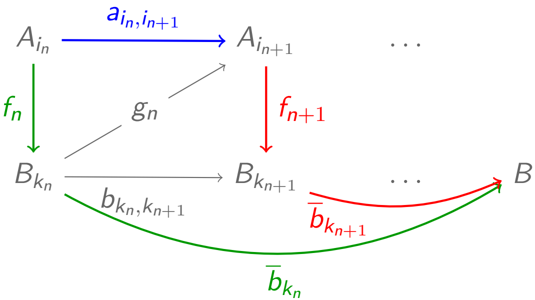

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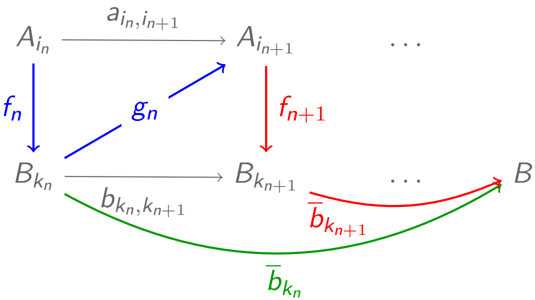

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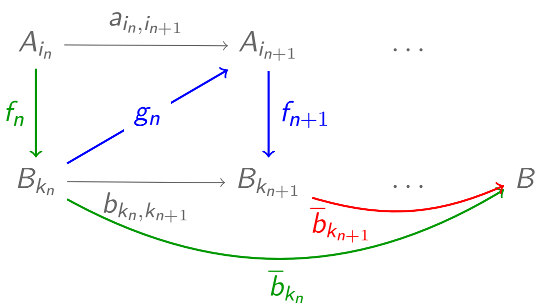

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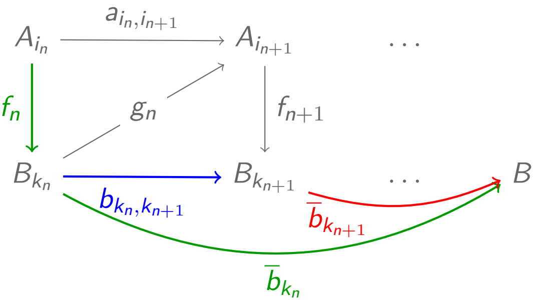

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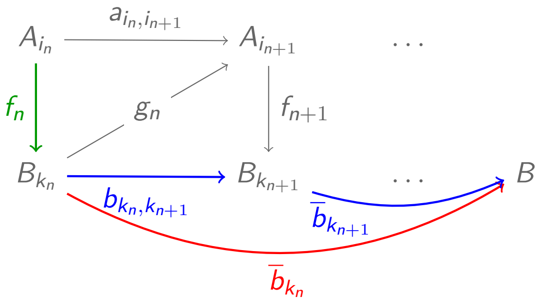

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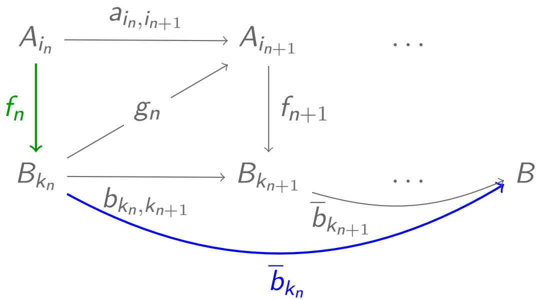

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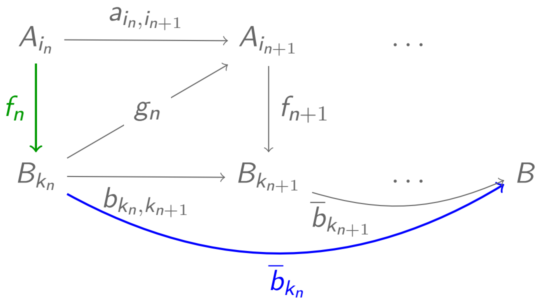

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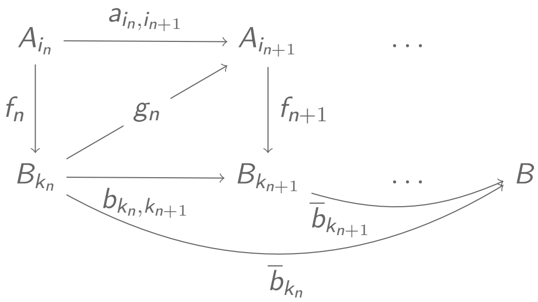

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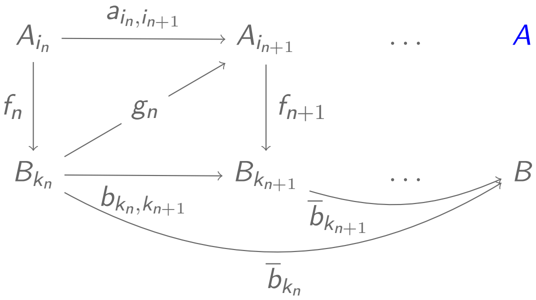

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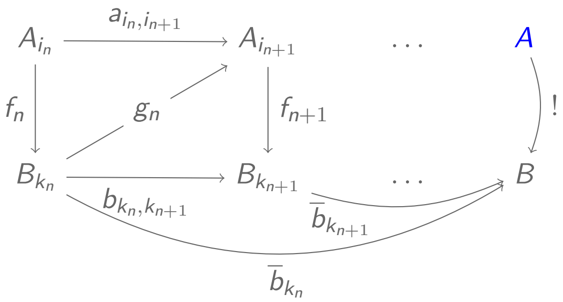

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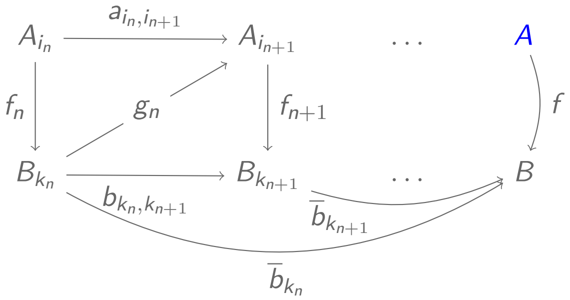

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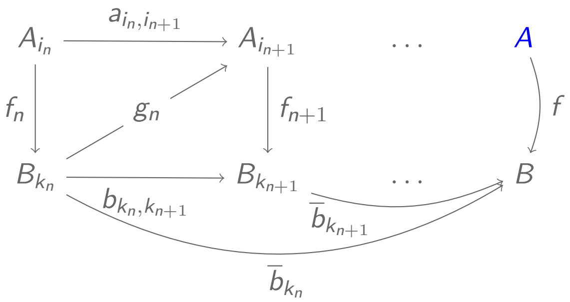

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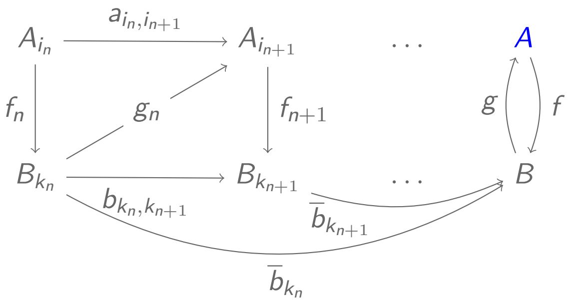

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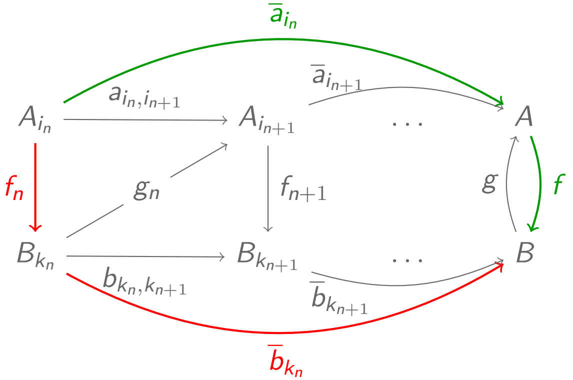
1. All cocone arrows factor through f i.e., $\overline{b}_{k_n} \circ f_n = f \circ \overline{a}_{i_n}$



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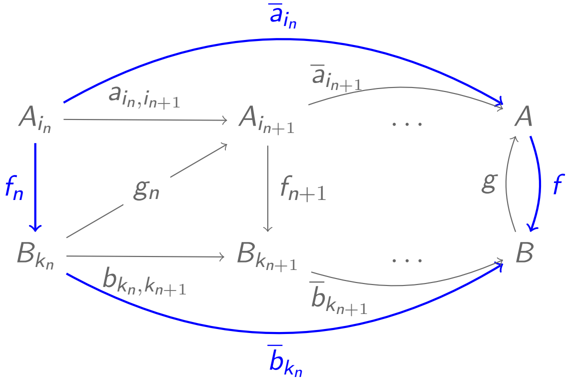

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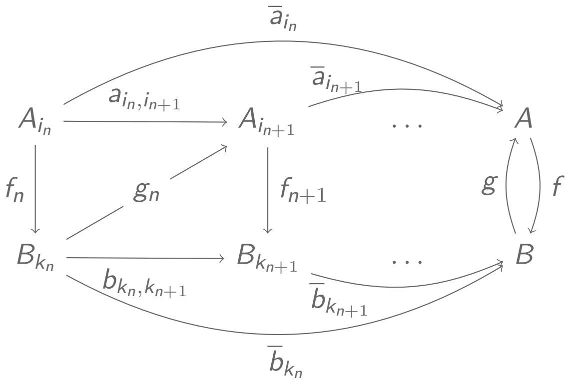

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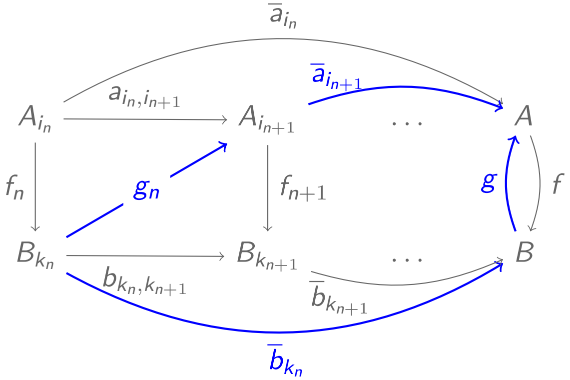

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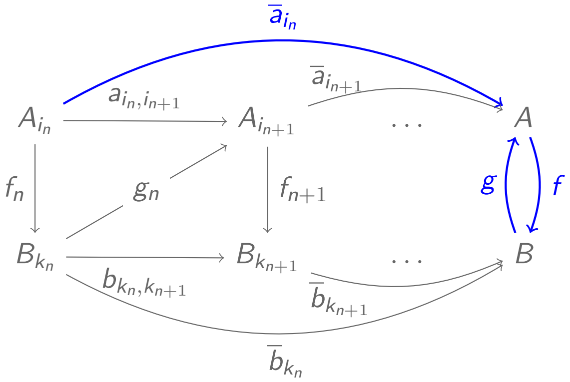
The composition $g \circ f$ is the identity



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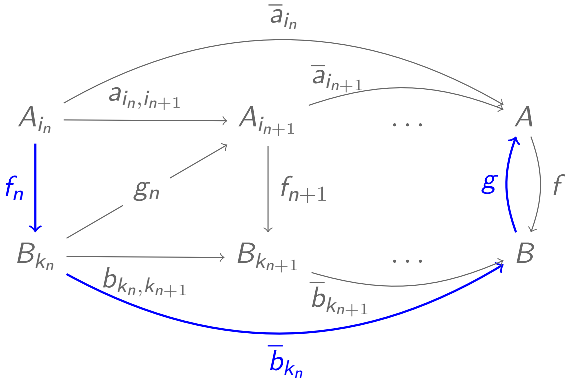

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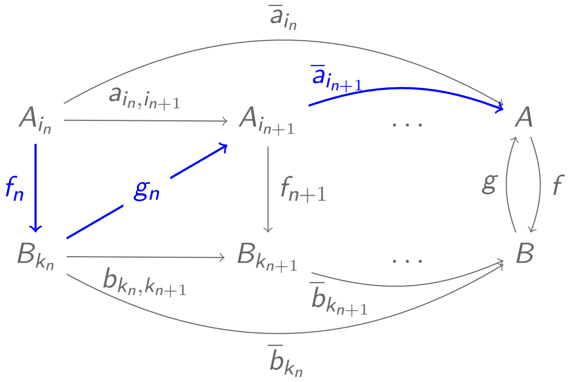

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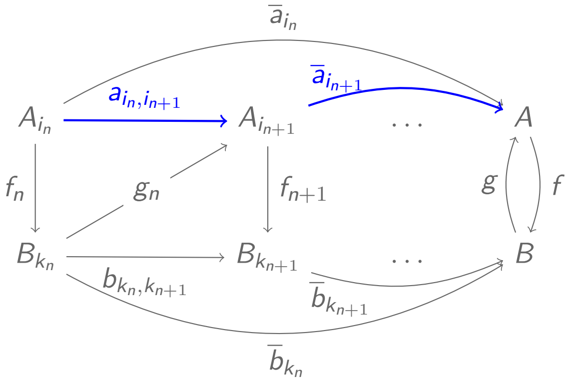

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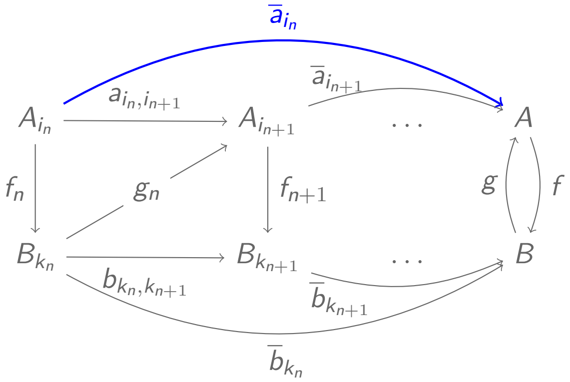

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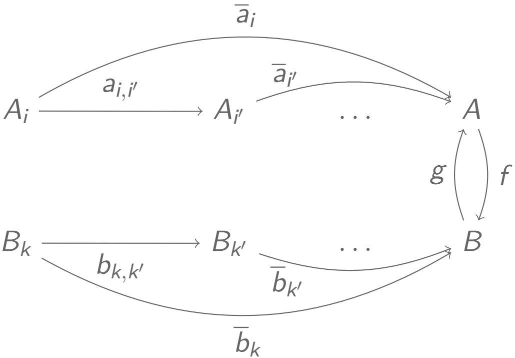

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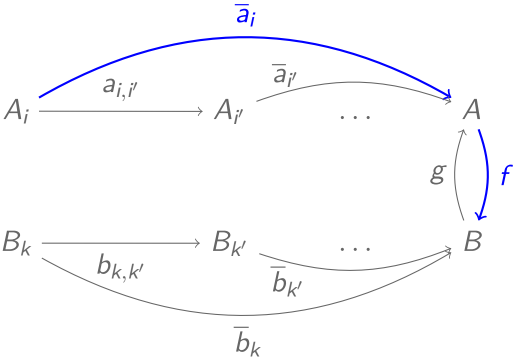

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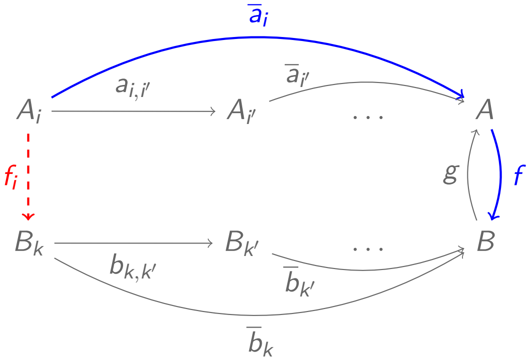

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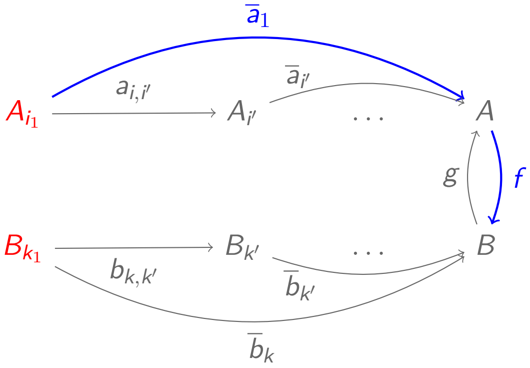

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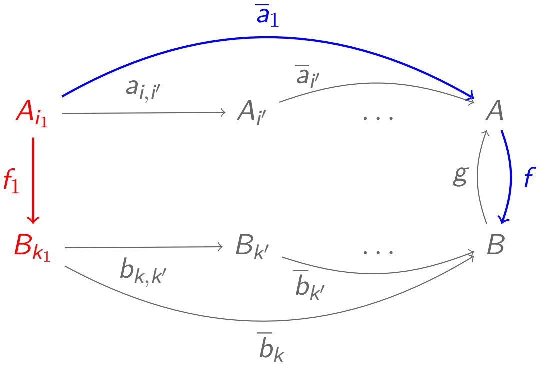

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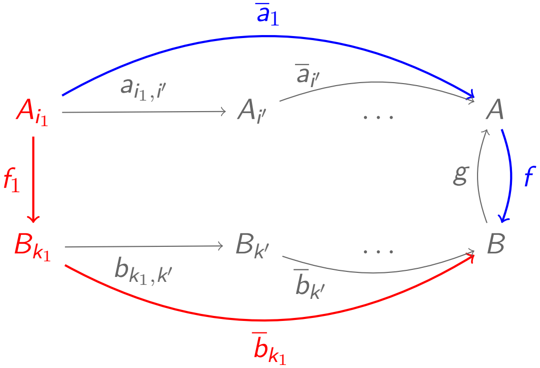

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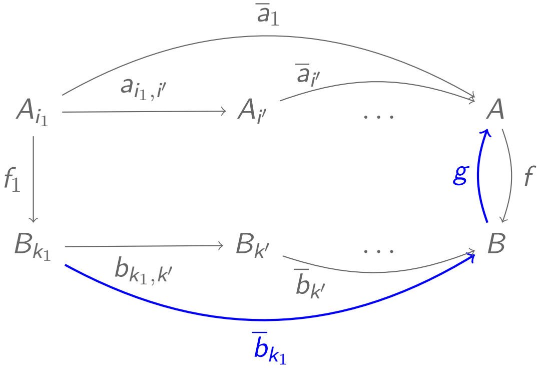

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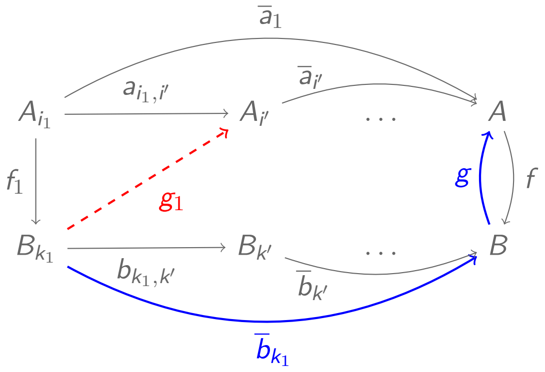

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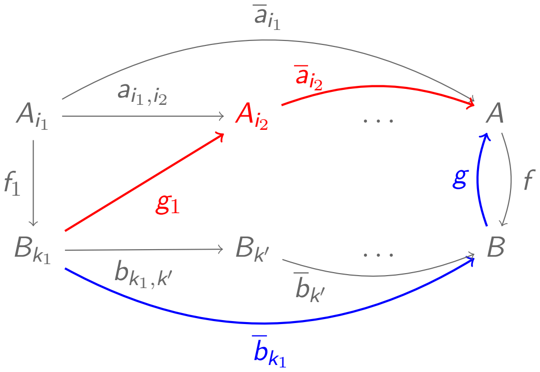

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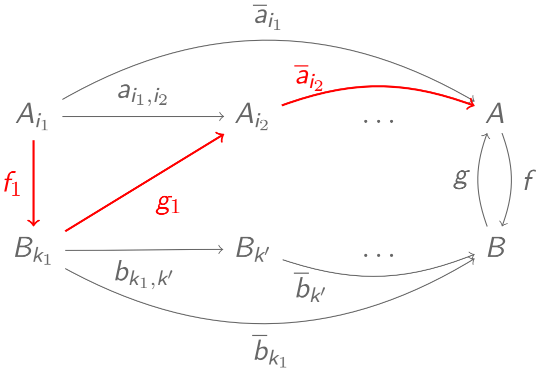

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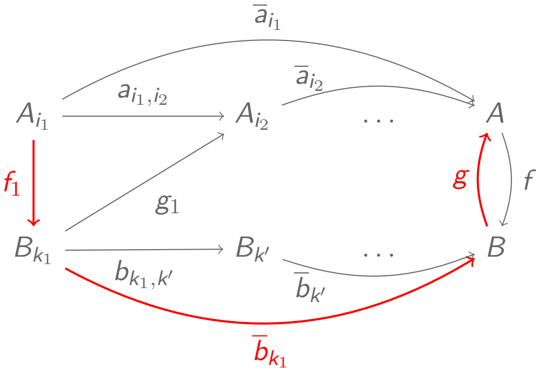

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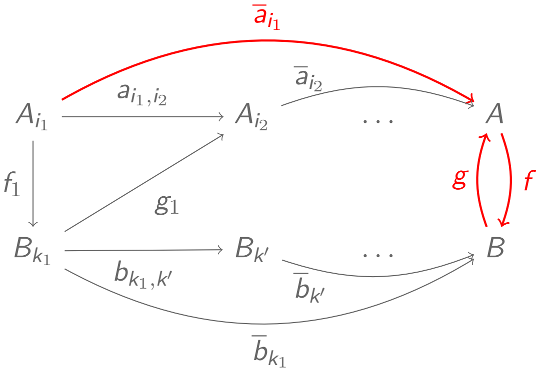

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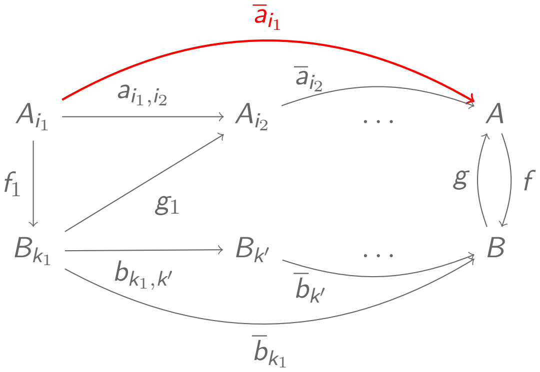

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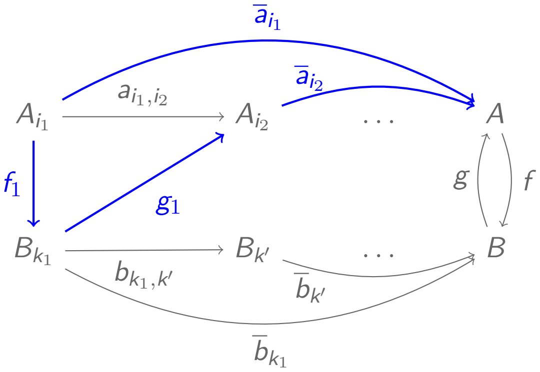

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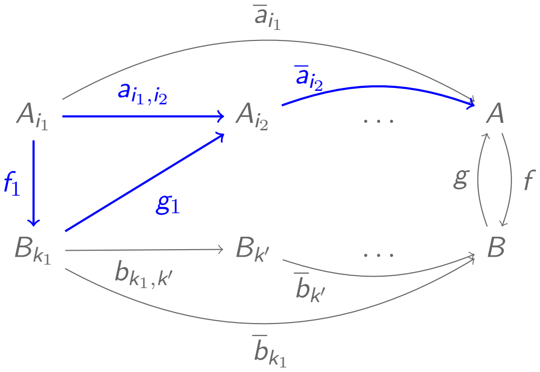

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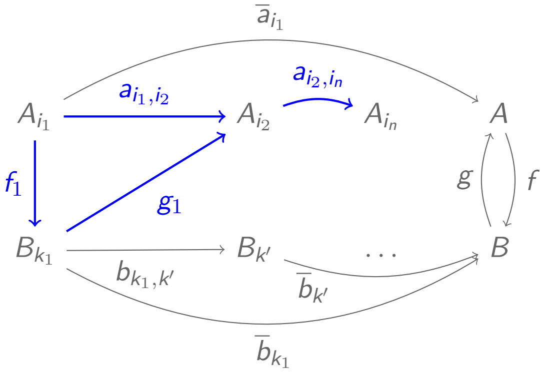

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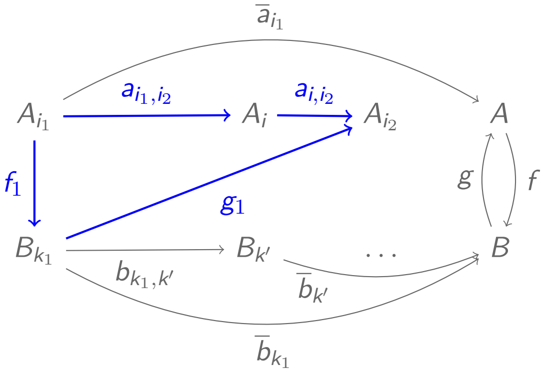

$u\ell$-groups

The main result of

- M.Busaniche, L. Cabrer, and D. Mundici. Confluence and combinatorics in finitely generated unital lattice-ordered abelian groups, *Forum Mathematicum*, 24 (2012), 253-271.

is as follows:

Theorem 3.3

Given direct systems S and T of finitely presented unital ℓ-groups with surjective connecting unital ℓ-homomorphisms let (G, u) and (H, v) denote their respective direct limits. Then the following conditions are equivalent:

1. $(G, u) \cong (H, v)$
2. S and T are confluent.

MV-algebras

Aslo in

- D. Mundici, Advanced Łukasiewicz calculus and MV-algebras, Trends in Logic, Vol. 35 Springer, New York, (2011).

Corollary 8.8

Let $D = A_0 \twoheadrightarrow A_1 \twoheadrightarrow A_2 ...$ and $E = B_0 \twoheadrightarrow B_1 \twoheadrightarrow B_2 ...$, be direct systems of finitely presented MV-algebras with surjective homomorphisms. Let A and B denote their respective direct limits. Then the following conditions are equivalent:

1. $A \cong B$.
2. D and E are confluent.

Torsion-free Abelian groups

Derry, Kurosh, and Mal'cev's isomorphisms invariants for countable torsion-free Abelian groups.

Torsion-free Abelian groups

Derry, Kurosh, and Mal'cev's isomorphisms invariants for countable torsion-free Abelian groups.

- Represent each countable torsion-free Abelian group as the directed colimit of some sequence

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Torsion-free Abelian groups

Derry, Kurosh, and Mal'cev's isomorphisms invariants for countable torsion-free Abelian groups.

- Represent each countable torsion-free Abelian group as the directed colimit of some sequence

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- Obtain an equivalence relation on sequences of matrices.

Dimension groups

The Bratteli-Elliott Isomorphism Criterion

Two sequences of simplicial groups and order-preserving group homomorphisms have isomorphic colimit dimension groups if, and only if, they are confluent.