The geometry of free MV_n algebras

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Motivations

- MV-algebra are the equivalent algebraic semantic of Łukasiewicz logic.
- MV-algebras are categorical equivalent to unital ℓ-groups.
- The free m-generated MV-algebra is the algebra of all piece-wise linear functions with integer coefficients from $[0, 1]^m$ to $[0, 1]$.
- Finding new characterizations of the free MV-algebras gives new insight in such a class.

An example of MV-algebra

The unit interval of real numbers $[0,1]$ endowed with the following operations:

$$x \oplus y = \min(1, x+y) \quad x \odot y = \max(0, x+y-1)$$
$$\neg x = 1 - x,$$

is an MV-algebra.

Theorem

The MV-algebra $\mathcal{S} = ([0,1], \oplus, \odot, \neg, 0, 1)$ generates the variety $\mathbb{MV}$, in symbols $\mathcal{V}(S) = \mathbb{MV}$.

Łukasiewicz logic with n+1 truth values

The subvarieties $\mathbb{MV}_n \subset \mathbb{MV}$ are axiomatized by the extra axiom: $x^{n+1} = x^n$ (or $(n+1)x = nx$).
The subvarieties $\mathbb{MV}_n$ corresponds to Łukasiewicz logic with n+1 truth values.
Let $\omega_0 := \omega \setminus \{0\}$. For $n \in \omega_0$ we set $\mathcal{S}_n = (S_n; \oplus, \odot, \neg, 0, 1)$, where

$$S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$$

and the operations $\oplus, \odot, \neg$ are defined as in S.
Then we have that $\mathbb{MV}_n = \mathcal{V}(\{S_1, ..., S_n\})$.

Free MV_n algebras

Let $F_{\mathbb{MV}_n}(m)$ be free m-generated MV-algebra in the variety $\mathbb{MV}_n$. Let $F_{\mathbb{MV}}(m)$ be free m-generated MV-algebra in the variety $\mathbb{MV}$.
Define the function $v_m(x)$ as follows:

$$\begin{aligned} v_m(1) &= 2^m, \\ v_m(2) &= 3^m - 2^m, \\ &\vdots \\ v_m(n) &= (n+1)^m - (v_m(n_1) + \ldots + v_m(n_{k-1})), \end{aligned}$$

where $n_1 = 1, n_k = n$ and $n_2, \ldots, n_{k-1}$ are the strict divisors of n.

Proposition ([1])

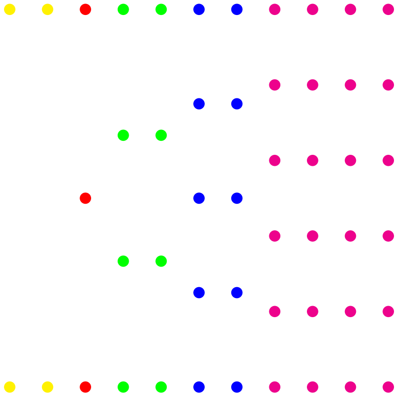
$$F_{\mathbb{MV}_n}(m) \cong S_1^{v_m(1)} \times \ldots \times S_n^{v_m(n)}.$$

[1] A. Di Nola , R. Grigolia, G. Panti, Finitely generated free MV-algebras and their automorphism groups, *Studia Logica*, **61**(1):65-78. 1998.

Some examples

The 1-generated case: $v(n) = (n+1) - (v(n_1) + ... + v(n_{k-1}))$
$v(1) = 2$, $v(2) = 3 - 2 = 1$, $v(3) = 4 - 2 = 2$, $v(4) = 5 - 2 - 1 = 2$,
$v(5) = 6 - 2 = 4$.

$F_{\mathbb{MV}_1} =$



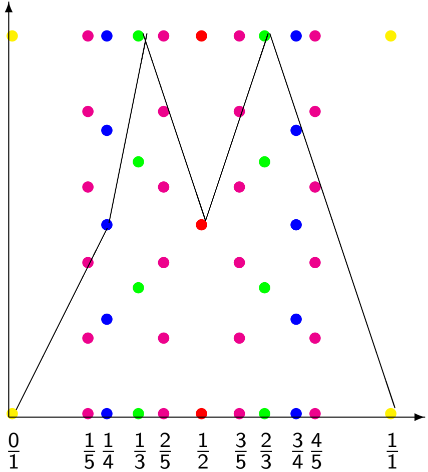
A characterization of the elements of F_{MV_n}

Proposition ([2])

Given a tuple $(a_1, ..., a_n)$ *in* $F_{\mathbb{MV}_k}$ *there is a McNaughton function* $f(x)$ *such that the set* $\{a_1, ..., a_n\}$ *is exactly the range of* $f(x)$ *restricted to* $\bigcup_{i=1}^{k} S_k$.

[2] A. Di Nola , R. Grigolia, G. Panti, Finitely generated free MV-algebras and their automorphism groups, *Studia Logica*, **61**(1):65-78. 1998.

An element of F_{MV_5}



Two different visualizations

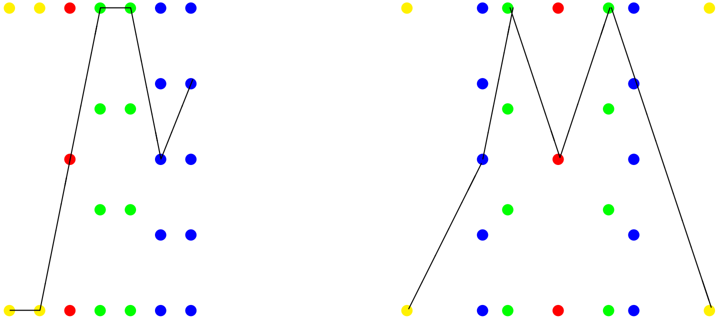


Figure: The black lines in the figures depict the same element of $F_{\mathbb{MV}_5}(1)(= S_1^2 \times S_2 \times S_3^2 \times S_4^2)$

Formalizing such visualizations

Definition

$\mathcal{Q}$ is the set of irreducible fractions between 0 and 1, endowed with the natural order, which we will indicate as usual with $<$.
$\mathcal{Q}^{\prec}$ has the same domain of $\mathcal{Q}$ but its linear order $\prec$ is given by

$$\frac{m}{n} \prec \frac{p}{q} \text{ if, and only if, } n < q \text{ or, if } n = q \text{ and } m < p$$

So the $\prec$-sorted listing of $\mathcal{Q}$ is $\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \ldots\}$.

The direct system

The direct limit is a categorical construction:

Definition

Let $(I, \leq)$ be a directed set. Let $\{A_i \mid i \in I\}$ be a family of objects indexed by I and suppose we have a family of embeddings $\varepsilon_{ij} : A_i \to A_j$ for all $i \leq j$ with the following properties:

- ε_{ii} is the identity in A_i,
- $\varepsilon_{ik} = \varepsilon_{jk} \circ \varepsilon_{ij}$ for all $i \leq j \leq k$.

Then the pair (A_i, ε_{ij}) is called a direct system over I.

The direct limit of a system

Definition

The underlying set of the direct limit, A, of the direct system (A_i, ε_{ij}) is defined as the disjoint union of the A_i's modulo a certain equivalence relation $\sim$:

$$A = \varinjlim A_i = \coprod_i A_i \Big/ \sim .$$

Where, if $x_i \in A_i$ and $x_j \in A_j$, then $x_i \sim x_j$ if there is some $k \in I$ such that $\varepsilon_{ik}(x_i) = \varepsilon_{jk}(x_j)$.
One naturally obtains from this definition canonical morphisms $\varphi_i : A_i \to A$ sending each element to its equivalence class. The algebraic operations on A are defined via these maps in the obvious manner.

The embeddings between F_{MV_n}

We now define a family of embeddings $\varepsilon_k : F_k \to F_{k+1}$.

- Given a tuple $(a_1, \ldots, a_n)$ in F_k we know that there is a McNaughton function $f(x)$ such that the set $\{a_1, \ldots, a_n\}$ is exactly the range of $f(x)$ restricted to $\bigcup_{i=1}^{k} S_k$.
- Define $\varepsilon(a_1, \ldots, a_n)$ as the tuple given by the domain of $f(x)$ when restricted to $\bigcup_{i=1}^{k+1} S_{k+1}$.

But, how to choose f?

- In the 1-generated let's just choose the *simplest*.

Formalizing the idea

Definition

Let us define for any $\frac{n}{m} \in \mathcal{Q}$

$$(\frac{n}{m})^{+} = \max\{\frac{a}{b} \in \mathcal{Q} \mid \frac{a}{b} < \frac{n}{m} \text{ and } b < m\}$$

and

$$(\frac{n}{m})^{-} = \min\{\frac{a}{b} \in \mathcal{Q} \mid \frac{a}{b} > \frac{n}{m} \text{ and } b < m\}.$$

Formalizing the idea (cont'd)

Definition

Let $a = (a_{\frac{0}{1}}, a_{\frac{1}{1}}, \ldots, a_{\frac{i}{k}})$ be, for a suitable $\frac{i}{k} \in \mathcal{Q}$, an element of F_k, then we define:

$$\varepsilon_k(a) = (a_{\frac{0}{1}}, a_{\frac{1}{1}}, \ldots, a_{\frac{i}{k}}, a_{\frac{1}{k+1}}, \ldots, a_{\frac{j}{k+1}})$$

where for all $\frac{j}{k+1} \in \mathcal{Q}$, we let $a_{\frac{j}{k+1}}$ be the solution of the linear equation:

$$\frac{\frac{j}{k+1} - (\frac{j}{k+1})^-}{(\frac{j}{k+1})^+ - (\frac{j}{k+1})^-} = \frac{a_{\frac{j}{k+1}} - a_{(\frac{j}{k+1})^-}}{a_{(\frac{j}{k+1})^+} - a_{(\frac{j}{k+1})^-}}$$

Lemma

ε_k is an embedding from F_k to F_{k+1}.

The snakes

Lemma (Characterization of the direct limit D)

For any element $a \in D$ there exists a unique $i \in \omega$ and a unique infinite sequences $(a^{(i)}, a^{(i+1)}, \dots)$ such that

(i) *for any $j \geq i$ there is exactly one $a^{(j)}$ in the sequence, such that $a^{(j)} \in F_j$;*

(ii) *$a^{(i)}$ has no inverse image with respect to ε_i;*

(iii) *$\varepsilon_{kj}(a^{(k)}) = a^{(j)}$ for any $k, j \geq i$:*

(iv) *for any $a^{(j)}$ in the sequence, the equivalence class of $a^{(j)}$ is a.*

Vice versa, given a sequence which satisfies the conditions $(i) - (iii)$ above there exists a unique $a \in D$ for which the condition (iv).

The snakes (cont'd)

Definition

Given any element $a \in D$ we will call the sequence given by the lemma above, the snake of a.

Lemma

For every snake $a = (a^{(i)}, a^{(i+1)}, \dots)$, there exists a unique McNaughton function $f(x)$ such that for any $k \geq i$ there exists $p \in \mathcal{Q}$ such that $a^{(k)} = (f(q))_{q \prec p}$

Lemma

Let $a, b \in D$ and let $(a^{(i)}, a^{(i+1)}, \dots)$ and $(b^{(j)}, b^{(j+1)}, \dots)$ their respective snakes. If $i \leq j$ then for some l the sub-sequence $(a^{(j+l)} \oplus b^{(j+l)}, a^{(j+l+1)} \oplus b^{(j+l+1)}, \dots)$ of $(a_j \oplus b_j, a_{j+1} \oplus b_{j+1}, \dots)$ is a snake.

Reconstructing the MV-algebra

Definition

We define the operation $\oplus$ in D as follows: let $a, b \in D$ and let $(a^{(i)}, a^{(i+1)}, a^{(i+2)}, \ldots)$ and $(b^{(j)}, b^{(j+1)}, b^{(j+2)} \ldots)$ their respective snakes then $a \oplus b$ is defined as the element of D whose snake is inside $(a^{(j)} \oplus b^{(j)}, a^{(j+1)} \oplus b^{(j+1)}, \ldots)$.

Theorem

The algebra $\langle D, \oplus, \odot, \neg, 0, 1\rangle$ is isomorphic to the MV-algebra $\langle M, \oplus, \odot, \neg, 0, 1\rangle$ of all McNaughton function in one variable.

Note that even if we used the symbol $\oplus$ we have not proved that $\langle D, \oplus, \odot, \neg, 0, 1\rangle$ is an MV-algebra. Indeed the proof of the above theorem directly shows that such a structures is isomorphic to the 1-generated free MV-algebra.

Summing up

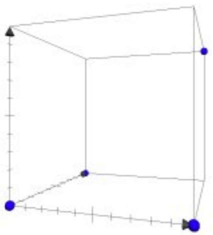
Summing up:

- We have defined an embedding between the free algebra in MV_n and the free algebra in MV_{n+1}.
- We have shown that the equivalence classes of such direct limit have peculiar properties which allow to put them in a bijective correspondence with McNaughton functions.
- Taking advantage of such a correspondence we define a operation which makes the direct limit an MV algebra.

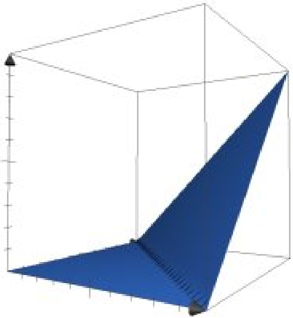
So the main issue is to define the embedding in such a way that this bijective correspondence is guaranteed. This was solved in the 1-generator case by *attaching* a McNaughton function to a tuple, in such a way that this is preserved under the embeddings.

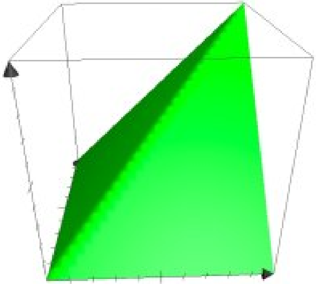
The problems encountered

The difficulties which arise in the general case lay on the fact that while the dimension increases polynomialy in the number of generators (n+1), the number of points to *interpolate* grows exponentially (2^n).



Two McN's functions associated to the same element





How to choose f?

The order $\prec^*$

Definition

The linear order $\langle \mathcal{Q}^n, \prec^* \rangle$ is defined as follow:

- $\mathcal{Q}^n$ is the subset of direct product $\mathcal{Q}^n$ given by those tuples whose elements have the same denominator or are 0 or 1.
- The order $\prec^*$ is inherited form $\prec$ as follows:

$$
\begin{aligned}
&(x_1, \ldots x_n) \prec^* y_1, \ldots, y_n \text{ if, and only if,} \\
&\qquad x_1, \ldots, x_n \prec y_1 \text{ or } \ldots \text{ or } x_1, \ldots, x_n \prec y_n \\
&\text{Or, if this is not the case, then} \qquad x_1 \prec y_1, \\
&\qquad \text{or if } x_1 = y_1 \text{ then} \qquad x_2 \prec y_2, \\
&\qquad \vdots \qquad \vdots \\
&\qquad \text{or if } x_1 = y_1, \ldots, x_{n-1} = y_{n-1} \text{ then} \qquad x_n \prec y_n.
\end{aligned}
$$

A $\prec^*$-listing of $\mathcal{Q}$

$$(0,0),$$
$$(0,1),(1,0),(1,1),$$
$$(0,\frac{1}{2}),(1,\frac{1}{2}),(\frac{1}{2},0),(\frac{1}{2},1),(\frac{1}{2},\frac{1}{2}),$$
$$(0,\frac{1}{3}),(1,\frac{1}{3}),(\frac{1}{3},0),(\frac{1}{3},1),(\frac{1}{3},\frac{1}{3}),(0,\frac{2}{3}),(1,\frac{2}{3}),(\frac{1}{3},\frac{2}{3}),(\frac{2}{3},0),(\frac{2}{3},1),\ldots$$

Since $\langle \mathcal{Q}^m, \prec^* \rangle$ is enumerable, discrete linear order with an initial point it is order-isomorphic to $\langle \mathbb{N}, < \rangle$,

Definition

Let $\tau_m : \mathbb{N} \to \mathcal{Q}^m$ be the (only) bijection preserving the order and let σ_m be its inverse.

Suppose that we have a tuple $(a_1, ...a_k) \in \mathcal{F}_n(m)$ and we want to extend it to a tuple of $\mathcal{F}_{n+1}(m)$, we need to find $v_m(n+1)$ new points.
Such points will have $\mathcal{Q}^m$-indexes $(0, ..., 0, \frac{1}{m}), (0, ..., 0, 1, \frac{1}{m}), ..., (1, ..., 1, \frac{1}{m}), ..., (\frac{m-1}{m}, ..., \frac{m-1}{m})$, think of those indexes as point on the hyper-plane $z = 0$. Each of those points has 2^m adjacent points with coordinates whose denominator is smaller than $m + 1$.

Definition

Given a tuple $(q_1, ..., q_m)$ of rational numbers we define the set of its adjacent points to be the set of tuples $\{((q_1)^\circ, ..., (q_m)^\circ) \mid \circ = +, -\}$

An obvious attempt...

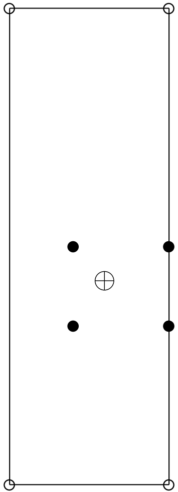
Definition

Let $(a_1, a_2, \ldots, a_{v_m(i)}) \in \mathcal{F}_i(m)$ and let $(q_1, \ldots, q_m) = \tau_m(j)$ for some $i < j < v_m(i+1)$. Let $p_1, \ldots, p_{2^m}$ be the adjacent points of $(q_1, \ldots, q_m)$. Let

$$\begin{array}{ll} \pi_1 \text{ be the hyper-plane s.t.} & (p_1, a_{\sigma_m(p_1)}), \ldots, (p_{m+1}, a_{\sigma_m(p_{m+1})}) \in \pi_1, \\ \pi_2 \text{ be the hyper-plane s.t.} & (p_2, a_{\sigma_m(p_2)}), \ldots, (p_{m+2}, a_{\sigma_m(p_{m+2})}) \in \pi_2 \\ \vdots & \vdots \\ \pi_i \text{ be the hyper-plane s.t.} & (p_i, a_{\sigma_m(p_i)}), \ldots, (p_{i+m+1}, a_{\sigma_m(p_{m+1})}) \in \pi_i. \end{array}$$

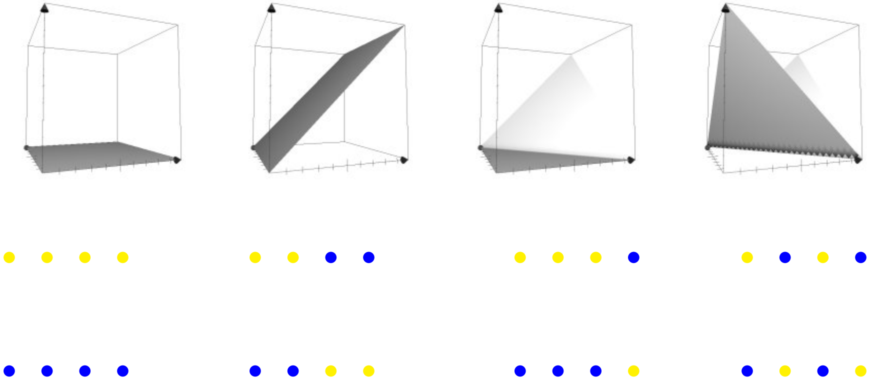
So we have $2^m - m$ (non necessarily distinct) hyper-planes. We call the curve associated to $(q_1, \ldots, q_m)$ the infimum π of all the hyper-planes $\pi_1, \ldots, \pi_{(2^m - m)}$ restricted to the polyhedron with vertexes in $\{(p_i, b) \mid 1 \leq i \leq 2^m \text{ and } b = 0, 1\}$.

... which does not work



$$\oplus = \left(\frac{3}{7}, \frac{5}{7}\right)$$

A 2-base



Conclusion

Conclusion

A corollary...

Corollary (Mundici)

A formula is valid in the Łukasiewicz infinite-valued calculus if, and only if, it is valid in all Łukasiewicz n-valued calculus for some n bigger of some given k.

...and a conjecture

Note that in the general case the construction has several steps in which arbitrary choses are made (e.g. when we chose to take the infimum of all functions associated to a tuple, or when we decided the way to arrange the elements of $\mathcal{F}_n(m)$ in the $m+1$-space). Call a construction *equivalent* if it satisfies the same lemmata as above.

Conjecture

An automorphism f of $\mathcal{F}(m)$ sends $a \in \mathcal{F}(m)$ in $f(a)$ iff there are two equivalent constructions which associate to a tuple in $\mathcal{F}_n(m)$ for some n the two different McNaughton functions given by a and $f(a)$.