Are locally finite MV-algebras a variety?

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The problem

Mundici's book *Advances Łukasiewicz calculus* ends with a list of eleven open problems.

Question (Problem n. 3)

Is the category of locally finite MV-algebras equivalent to an equational class?

Answer

It depends!

The complete answer

Answer

1. The category of locally finite MV-algebras is not equivalent to any finitary variety.
2. More is true: the category of locally finite MV-algebras is not equivalent to any finitely-sorted finitary quasi-variety.
3. The category of locally finite MV-algebras is equivalent to an infinitary variety.
4. The category of locally finite MV-algebras is equivalent to a countably-sorted finitary variety.

Overview

To prove the result we use:

1. The duality between locally finite MV-algebras and the category of multisets proved by Cignoli, Dubuc and Mundici.
2. A characterisation of finitary quasi-varieties as co-complete categories that have an abstractly finite, regular projective regular generator.
3. A characterisation of infinitary quasi-varieties as co-complete categories that have a regular projective regular generator.
4. A characterisation of varieties (finitary or infinitary) as quasi-varieties in which all internal equivalence relations are effective.

Historical remarks

The last three characterisations belong to a rich research stream that started in the 60's with the work of Bénabou, Diers, Isbell, Lawvere, Linton, Wraith and then continued by Adámek, Pedicchio, Rosicky, Vitale, Wood, and many others.

The characterisations above are essentially due to Isbell, but the improved versions we use are due to Adámek. The work of many others is implicitly used to come to further simplifications.

Locally finite MV-algebras

Recall that an algebra A is called **locally finite** if every finitely generated subalgebra of A is finite.

Let $\mathsf{MV}_{\mathsf{lf}}$ be the category of locally finite MV-algebras with MV-homomorphisms between them.

Locally finite MV-algebras

Theorem

For every MV-algebra A the following conditions are equivalent:

1. A is locally finite.
2. A is the direct limit (colimit) of a direct system $\{(A_i, \iota_{ij}) \mid i, j \in J, i \leq j\}$ of finite MV-algebras with injective homomorphisms $\iota_{ij} \colon A_i \to A_j$.
3. For each prime ideal P of A, A/P is isomorphic to a subalgebra of $\mathbb{Q} \cap [0, 1]$.
4. For some Stone space Y, A is isomorphic to a separating subalgebra of the MV-algebra $C_{\mathbb{Q}}(Y)$ consisting of functions of finite range.

Remark: $\mathsf{MV}_{\mathsf{lf}} \simeq \text{ind-}\mathsf{MV}_{\mathsf{finite}}$

Cignoli-Dubuc-Mundici's duality, explained

Recall that

$$\mathsf{BA} \simeq \text{ind-}(\mathsf{BA}_{\mathsf{finite}}) \simeq \text{ind-}({\mathsf{Set}_{\mathsf{finite}}}^{\mathrm{op}}) \simeq (\text{pro-}(\mathsf{Set}_{\mathsf{finite}}))^{\mathrm{op}} \simeq \mathsf{Stone}^{\mathrm{op}}.$$

$$\mathsf{MV}_{\mathsf{lf}} \simeq \text{ind-}(\mathsf{MV}_{\mathsf{finite}}) \simeq \text{ind-}({\mathsf{MultiSet}_{\mathsf{finite}}}^{\mathrm{op}}) \simeq (\text{pro-}(\mathsf{MultiSet}_{\mathsf{finite}}))^{\mathrm{op}}$$
$$\simeq\ ??^{\mathrm{op}}.$$

Supernatural numbers

Definition

A **supernatural number** is a function

$$\nu \colon \mathbb{P} \longrightarrow \{0, 1, 2, \ldots, \infty\}.$$

The set of supernatural numbers forms a complete lattice under point-wise order.

A supernatural number ν is said **finite** iff

- ∞ does not belong to the range of ν,
- $\nu(p)$ is not zero only for a finite number of p.

One-one corresp. $n \leftrightarrow \nu_n$ between $\mathbb{N}_{>0}$ and finite elements of $\mathcal{N}$.

E.g., $$\nu_{12}(p) = \begin{cases} 2 & \text{if } p = 2 \\ 1 & \text{if } p = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Supernatural numbers

The set $\mathcal{N}$ is equipped with the topology having as an open basis

$$U_n = \{\nu \in \mathcal{N} \mid \nu \geq \nu_n\}, \text{ for } n \in \mathbb{N}_{>0}.$$

A sub-basis for this topology is given by the sets

$$U_{p,m} = \{\nu \in \mathcal{N} \mid \nu(p) > m\}, \text{ for } p \in \mathbb{P} \text{ and } m \in \mathbb{N}.$$

The above-described topology coincides with the Scott topology.

Multisets

Definition

The category MS of multisets.
Objects: a **multiset** is a pair (X, ζ), where X is a Stone space, and $\zeta\colon X \to \mathcal{N}$ is continuous. The map ζ is called the **denominator map**.

Arrows: a continuous function $f\colon (X, \zeta_X) \to (Y, \zeta_Y)$ that **respects denominators** i.e., for every $x \in X$,

$$\zeta_X(x) \geq \zeta_Y(f(x)).$$

Theorem (Cignoli, Dubuc, Mundici 2004)

The category $\mathsf{MV}_{\mathsf{lf}}$ is equivalent to the category $\mathsf{MS}^{\mathrm{op}}$.

A characterisation of infinitary quasi-variety

Theorem

A (locally small) category C is equivalent to an infinitary (single-sorted) quasi-variety of algebras

if, and only if,

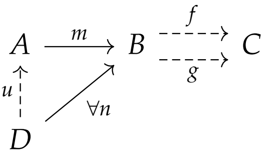
C is co-complete and has a regular projective regular generator.

Remark

For multi-sorted theories, one replaces "regular generator" with regular generating set of objects.

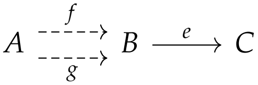
Regularity

An arrow $m\colon A \to B$ is **regular monic** if there exists a pair of parallel arrows $f, g\colon B \to C$ such that m is their equaliser, i.e.,



$$f \circ m = g \circ m$$

Dually, an arrow $e\colon B \to C$ is **regular epic** if there exists a pair of parallel arrows $f, g\colon A \to B$ such that e is their co-equaliser, i.e.,



Generators

In a co-complete category, a set $\mathcal{G} = \{G_s \mid s \in S\}$ of objects is called a **set of generators** if for every object A, the canonical quotient

$$\sum_{s \in S} \sum_{\hom_{\mathbf{C}}(G_s, A)} G_s \to A \text{ is epic.}$$

A set $\mathcal{G}$ is **regularly generating** if for every object A, the canonical quotient

$$\sum_{s \in S} \sum_{\hom_{\mathbf{C}}(G_s, A)} G_s \to A \text{ is regular epic.}$$

An object G is a **regular generator** if for every object A, the canonical arrow

$$\sum_{\hom_{\mathbf{C}}(G, A)} G \to A \text{ is regular epic.}$$

Examples

Example

In the category Set, finite sets form a set of generators.

Example

If $\mathbb{V}$ is a variety, finitely presented algebras form a set of generators.

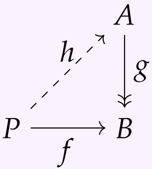
Example

If $\mathbb{V}$ is a variety, $\mathcal{F}_{\mathbb{V}}(1)$ is a generator.

Regular projective objects

Definition

An object P is called **regular projective** if for any arrow $f\colon P \to B$ and every regular epic arrow $g\colon A \twoheadrightarrow B$, the arrow f factors through g, i.e., there exists h such that the following diagram commutes.



A characterisation of infinitary quasi-variety

Theorem

A (locally small) category C is equivalent to an **infinitary quasi-variety** of algebras

if, and only if,

C is co-complete and has a regular projective regular generator.

To use this theorem we will see that MS is a complete category having a regular co-generating regular injective object.

The category MS

We start with an observation which will simplifies calculations in MS.

Theorem

The forgetful functor $U\colon \mathsf{MS} \to \mathsf{Stone}$ is topological.

Corollary

The forgetful functor $U\colon \mathsf{MS} \to \mathsf{Stone}$ has both a left and a right adjoint both of which are full embeddings.

Corollary

The category of multisets is complete and co-complete.

The category MS

Corollary

Let $f : X \to Y$ be an arrow in MS.

1. f is epic if, and only if, it is surjective.
2. f is monic if, and only if, it is injective.
3. f is regular monic if, and only if, it is injective and for every $x \in X$, we have $\zeta(x) = \zeta(f(x))$. [**preserves denominators**]
4. f is an iso if, and only if, f is bijective and, for every $x \in X$, we have $\zeta(x) = \zeta(f(x))$.

The multisets $\mathbb{2}$ and $\mathbb{D}_n$

For any $n \in \mathbb{N}_{>0}$ let

$$\mathbb{D}_n := (\{0,1\}, \zeta_n)$$

where the topology is discrete and

$$\zeta_n(0) := \nu_1 \text{ and } \zeta_n(1) := \nu_n.$$

Set $\mathbb{2} := \mathbb{D}_1$, notice that $\zeta_{\mathbb{2}}(0) = \zeta_{\mathbb{2}}(1) = \nu_1$.

Co-genereting sets in MS

Lemma

A set of objects $\mathcal{K}$ in MS is co-generating if, and only if, there exists $G \in \mathcal{K}$ that has at least two distinct points of denominator ν_1.

Sketch.

($\Rightarrow$) The canonical arrow $f : \mathbb{2} \to \prod_{G \in \mathcal{G}} \prod_{\hom(\mathbb{2},G)} G$ is monic. Therefore, there exists $G \in \mathcal{G}$ and $t \in \hom(\mathbb{2}, G)$ which is monic.

($\Leftarrow$) Suppose there exists $G \in \mathcal{K}$ with at least two distinct points of denominator ν_1. Then, there is a monic arrow $t : \mathbb{2} \to G$. But the discrete two-points space is a co-generator in Stone, hence $\mathbb{2}$ is a co-generator in MS. Since $t : \mathbb{2} \to G$ is monic, G is a co-generator, as well. □

Regular co-genereting sets in MS

Lemma

A set of objects $\mathcal{K}$ in MS is regular co-generating if, and only if, for every $p \in \mathbb{P}$ and $k \in \mathbb{N}$ there exists $G \in \mathcal{K}$ that has at least one point of denominator ν_1 and one point of denominator ν_{p^k}.

The proof idea is similar to the one of the previous lemma. Additionally, we use here that the elements ν_{p^k} are the completely join irreducible members of $\mathcal{N}$.

Injective multisets

Lemma

Let (X, ζ) be a multiset and suppose that the following conditions hold.

1. The set X is finite.
2. For every $x \in X$, $\zeta(x)$ is finite.
3. There exists an element in X with denominator ν_1.

Then (X, ζ) is regular injective in MS.

Corollary

For every $n \in \mathbb{N}_{>0}$, $\mathbb{D}_n$ is regular injective.

$\mathsf{MV}_{\mathsf{lf}}$ IS A QUASI-VARIETY OF ALGEBRAS

Corollary

The category $\mathsf{MV}_{\mathsf{lf}}$ is equivalent to an **infinitary quasi-variety of algebras**.

Proof.

Let

$$\mathcal{M} := \mathbb{2} \times \prod_{p \in \mathbb{P}, k \in \mathbb{N}} \mathbb{D}_{p^k}.$$

Since the product of regular injective objects is again regular injective, $\mathcal{M}$ is a regular injective. Notice that $\mathcal{M}$ has two points of denominator ν_1 and one point of denominator ν_n for every $n \in \mathbb{N}_{>0}$. So, MS is a regular co-generator. □

Abstractly finiteness

Definition

An object G is called **abstractly finite** if every arrow from G to a co-power of G factors through a finite sub-co-power.

Lemma (Pedicchio and Vitale 2000)

If G is an abstractly finite, regular projective, regular generator, then G is finitely generated. Vice versa, if G is finitely generated and has copowers, then G is abstractly finite.

A characterisation of finitary quasi-varieties

Theorem

A (locally small) category C is equivalent to an **finitary quasi-variety** of algebras

if, and only if,

C is co-complete and has a abstractly finite, regular projective regular generator.

$\mathsf{MV}_{\mathsf{lf}}$ IS NOT A FINITARY QUASI-VARIETY

Lemma

Finitely co-generated multisets are finite.

Theorem

The category $\mathsf{MV}_{\mathsf{lf}}$ is not equivalent to any finitary quasi-variety of algebras.

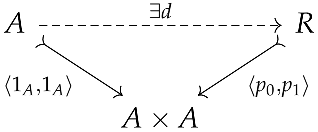
Proof.

A regular co-generating multiset must be infinite because it needs points of any possible denominator. □

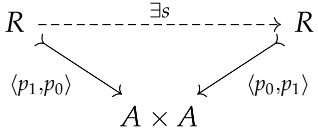
Internal equivalence relations

Let A be an object of C. An **(internal) equivalence relation** on A is a subobject $\langle p_0, p_1 \rangle \colon R \rightarrowtail A \times A$ satisfying:

reflexivity there exists an arrow $d \colon A \to R$ in C such that the following diagram commutes;

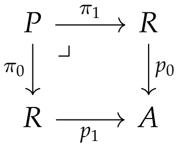


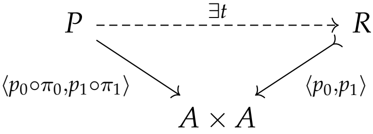
symmetry there exists an arrow $s \colon R \to R$ in C such that the following diagram commutes;



Effective equivalence relations

transitivity if the left-hand diagram below is a pullback square in C, then there is an arrow $t\colon P \to R$ such that the right-hand diagram commutes.





Effective equivalence relations

Definition

An equivalence relation $\langle p_0, p_1 \rangle \colon R \rightarrowtail A \times A$ is **effective** if there exists an arrow $q \colon A \to S$ such that $\langle p_0, p_1 \rangle \colon R \rightarrowtail A \times A$ is the kernel pair of q.

$$\begin{array}{ccc} R & \xrightarrow{p_1} & A \\ {\scriptstyle p_0}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ A & \xrightarrow[q]{} & S \end{array}$$

If C has co-equalisers, then an equivalence relation in C is effective if, and only if, it is the kernel pair of its co-equaliser.

For varieties of algebras, every equivalence relation is effective and they coincide with congruences.

A characterisation of infinitary varieties

Theorem

A (locally small) category C is equivalent to an **infinitary variety** of algebras

if, and only if,

1. C is equivalent to an infinitary quasi-variety of algebras,
2. all (internal) equivalence relations in C are effective.

Dual equivalence relations

To prove that in $\mathsf{MV}_{\mathsf{lf}}$ every equivalence relation is effective, we work again in the dual with **co-relations**, i.e., quotients.

They can be seen as pairs $(\sim, \mu)$, such that

1. $\sim$ is a Stone equivalence relation on X,
2. $\mu\colon X \to \mathcal{N}$ is a continuous function such that $\mu \leq \zeta$ and,
3. for all $x, y \in X$, if $x \sim y$, then $\mu(x) = \mu(y)$.

Reflexive co-relations

Theorem

$\mathsf{MV}_{\mathsf{lf}}$ is a Mal'cev category, i.e., every **reflexive relation** is an **effective equivalence relation**.

Theorem

$\mathsf{MV}_{\mathsf{lf}}$ is equivalent to an infinitary variety of algebras (with arity at most $\aleph_0$).

Thank you!