Fixed points in many-valued logic

Adding fixed point connectives to the most important t-norm based logics

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Overview

Motivations

Why introducing fixed points in many-valued logic?

- From a logical perspective, the high number of important results found in the cases of first order logic with fixed points as well as in modal logic with fixed points (μ-calculus).
- From the point of view of applications, fixed points stand at the heart of computer science. Having a formal system to threat them in a “fuzzy” way may rise new topics in approximate reasoning.
- From the algebraic side, fixed points enrich the structures of the algebraic semantics of the logic under study, leading to new and interesting structures.

Methods

How to give a "meaning" to the newly introduced fixed points ?

- The non-trivial part, when introducing fixed points in a logical systems, comes when looking for a semantics for such fixed points.
- The *classical* approach is based on Tarski theorem (see First order logic with fixed points, μ-calculus).
- In many-valued logic it is possible to use a new approach, based on continuity.

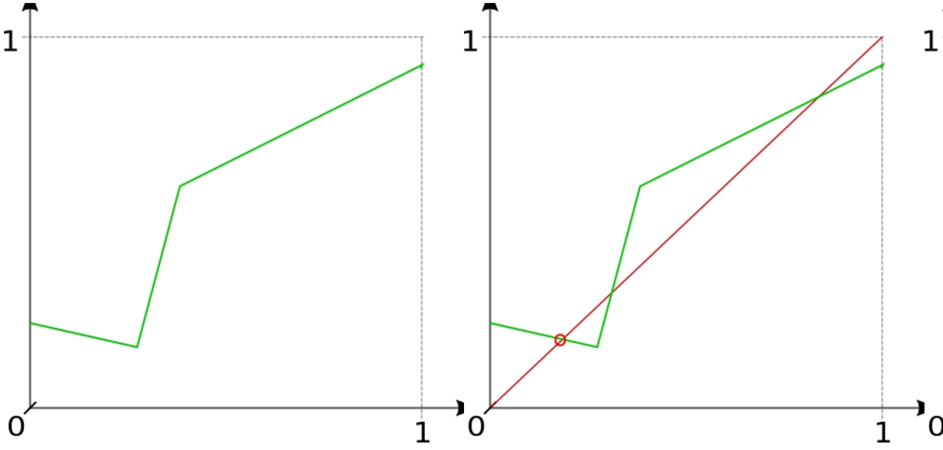
Methods

Theorem (Brouwer, 1909)

Every continuous function from the closed unit cube $[0,1]^n$ to itself has a fixed point.

- With this approach the kinds of formula which have fixed points are different, compared to the classical cases where one has to restrict to formulas on which the variable under the scope of μ only appears positively.
- On the other hand, the function giving the fixed point of a formula does not need to be continuous in the remaining variables, whence we can not allow nested occurences of μ.

Discontinuity of μ-functions



T-norm based logics

Definition

A t-norm $*$ is a function from $[0,1]^2$ to $[0,1]$ which is:

- associative: $x(y*z) = (x*y)*z$,
- commutative: $x*y = y*x$,
- non-decreasing: $x \leq y$ implies $x*z \leq y*z$,
- $x*1 = x$ and $x*0 = 0$,

Definition

A residual $\rightarrow$, of a t-norm $*$, is a function from $[0,1]^2$ to $[0,1]$ such that

$$x*y \leq z \text{ if, and only if, } x \leq y \rightarrow z$$

T-norm based logics

Definition

Gödel logic is the logic complete w.r.t the following connectives:

$$x \wedge y = \min\{x, y\} \qquad x \to_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Product logic is the logic complete w.r.t. the following connectives:

$$x \cdot y = xy \qquad x \to_\Pi y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$$

Łukasiewicz logic is the logic complete w.r.t:

$$x \oplus y = \min\{x + y, 1\} \qquad \neg x = 1 - x$$

Mostert and Shields' Theorem

The tree above logics are the most important logics based on continuous t-norm because of the following result:

Theorem (Mostert, Shields '57)

Every continuous t-norm is locally isomorphic to either Łukasiewicz, product or Gödel t-norm.

Definition

Basic logic (BL for short) is the logic complete w.r.t all continuos t-norms

Algebraic semantics

Theorem (Chang '58)

The algebraic semantic of Łukasiewicz logic is given by MV-algebras.

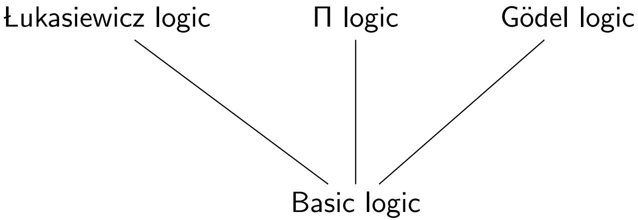
Theorem (Hájek '98)

The algebraic semantic of product logic is given by Π-algebras.

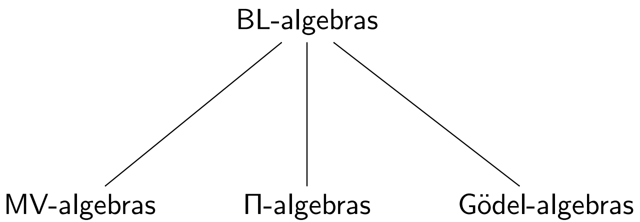
Theorem (Hájek '98)

The algebraic semantic of Basic Logic is given by BL-algebras.

Relations among t-norm based logics



Relations among algebraic semantics



ŁΠ logic

Definition

The logic ŁΠ is the logic having as set of primitive connectives $\{\oplus, \neg, \cdot, \to_\Pi, 0, 1\}$ satifying the following axioms:

- all axioms of Łukasiewicz logic for $\{\oplus, \neg, 0, 1\}$,
- all axioms of product logic for $\{\cdot, \to_\Pi, 0, 1\}$,
- $\varphi \cdot (\psi \ominus \xi) \leftrightarrow (\varphi \cdot \psi) \ominus (\varphi \cdot \xi)$,
- $\Delta(\varphi \to \psi) \to \varphi \to_\Pi \psi$.

Where $\Delta(\varphi)$ is defined as $(\neg\varphi) \to_\Pi 0$.
The rules are modus ponens and necessitation:

- If φ and $\varphi \to \psi$ then ψ,
- if φ then $\Delta(\varphi)$.

Importance of ŁΠ logic

ŁΠ logic has a stronger expressive power than the above mentioned logics, indeed:

Theorem (Esteva, Godo, Montagna '01)

ŁΠ logic faithfully interprets Łukasiewicz, product and Gödel logic. Moreover, if limited to finite deductions, also Pavelka logic is interpretable in ŁΠ logic.

More generally:

Theorem (Marchioni, Montagna '06)

Every logic based on a continuous t-norm with a finite number of idempotents is definable in ŁΠ logic.

ŁΠ algebras

Definition

ŁΠ algebras are the algebraic semantics of ŁΠ logic, so they are structures of type $\mathcal{A} = \langle A, \oplus, \neg, \cdot, \rightarrow_\Pi, 0, 1\rangle$.

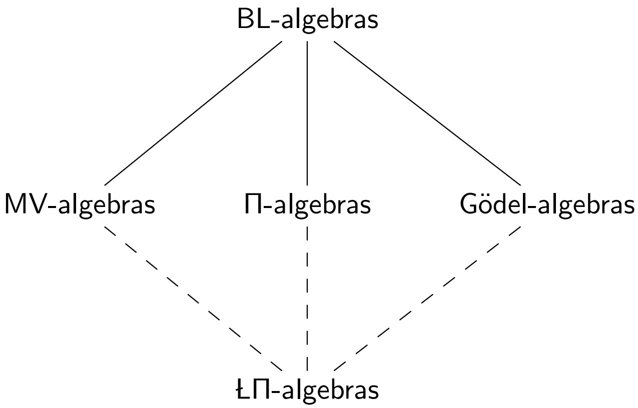
Example

The algebra $\langle [0,1], \oplus, \neg, \cdot, \rightarrow_\Pi, 0, 1\rangle$, where the operations are defined as follows:

- $x \oplus y = \min\{x + y, 1\}$ $\qquad \neg x = 1 - x$
- $x \cdot y = xy$ (ordinary product between reals)
- $x \rightarrow_\Pi y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$

is a ŁΠ-algebra. Moreover it generates the variety of ŁΠ-algebras.

Relations among algebraic semantics



μŁΠ logic

We introduce now the logic ŁΠ with fixed points.

Definition

The logic μŁΠ is obtained as an expansion of ŁΠ logic with a new (generalized) connective

$$\mu p_{\varphi(p,\bar{q})}(\bar{q})$$

for any ŁΠ-formula $\varphi(p, \bar{q})$ in which the symbol $\to_{\Pi}$ does not appear.

Such connectives must satisfy a number of axioms which we give directly in their algebraic form.

μŁΠ algebras

Definition

Let us call *CTerm* the set of ŁΠ terms in which the symbol $\to_\Pi$ does not appear. μŁΠ-algebras are structures of type

$$\mathcal{A} = \langle A, \oplus, \neg, \cdot, \to_\Pi, 0, 1, \{\mu x_{t(x,\bar{y})}\}_{t(x,\bar{y}) \in CTerm} \rangle$$

which satisfy the following axioms:

1. $\langle A, \oplus, \neg, \cdot, \to_\Pi, 0, 1 \rangle$ is a ŁΠ-algebra
2. $\mu x_{t(x,\bar{y})}(\bar{y}) = t(\mu x_{t(x,\bar{y})}(\bar{y}), (\bar{y}))$,
3. If $t(s(\bar{y}), \bar{y}) = s(\bar{y})$ then $\mu x_{t(x,\bar{y})}(\bar{y}) \leq s(\bar{y})$,
4. $\bigwedge_{i \leq n} \Delta(p_i \leftrightarrow q_i) \leq (\mu x_{t(x,\bar{y})}(p_1, \ldots, p_n) \leftrightarrow \mu x_{t(x,\bar{y})}(q_1, \ldots, q_n))$

Properties of μŁΠ-algebras

Theorem

The class of μŁΠ-algebra is a variety.

Theorem

The variety of μŁΠ-algebra is generated by its linearly ordered members.

Maximum fixed points

One may ask why only minimum fixed points are considered.

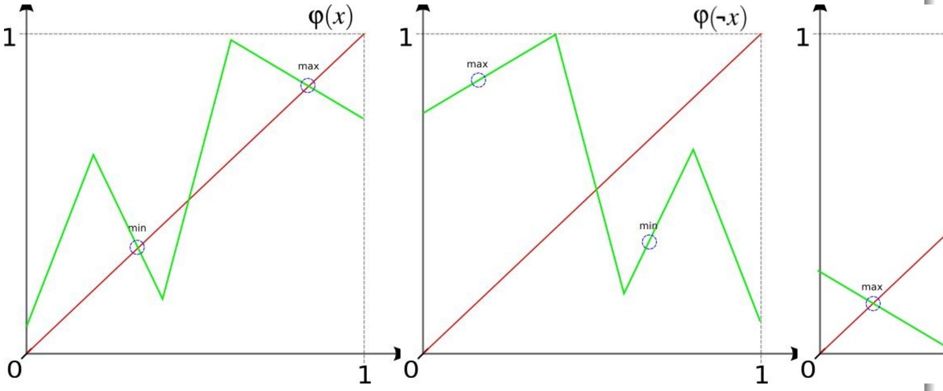
Theorem

For every $\varphi(x) \in Cterm$, the term defined by $\nu x_\varphi =^{def} \neg\mu x_{(\neg\varphi(\neg x))}$ has the following properties:

- $\varphi(\nu x_{\varphi(x)}) = \nu x_{\varphi(x)}$
- *If* $\varphi(t) = t$ *then* $t \to \nu x_{\varphi(x)}$

Hence it interprets the maximum fixed point of $\varphi(x)$.

Proof.



μŁΠ-algebra on [0,1]

Example

The algebra $\langle [0,1], \oplus, \neg, \cdot, \rightarrow_{\Pi}, 0, 1, \{\mu x_{t(x,\bar{y})}(\bar{y})\}_{t(x,\bar{y}) \in CTerm} \rangle$ is a μŁΠ-algebra.
Where $\mu x_{t(x,\bar{y})}(\bar{y})$ is the function which assigns to the tuple $\bar{a} \in [0,1]^n$ the minimum fixed point of the function $t(x,\bar{a})$ for each $t(x,\bar{a}) \in \mathbb{R}^{alg}[x]$.

Theorem

The μŁΠ-algebra on $[0,1]$ defined above generates the variety of μŁΠ-algebras.

μŁΠ-algebras and ordered fields

Definition

Given a linearly ordered Ł$\Pi\frac{1}{2}$ algebra $\mathcal{A}$, consider the structure $\Phi(\mathcal{A}) = \langle K, +, -, \times, \leq, 0_K, 1_K \rangle$ defined in the following way:

$$K = \{(z, x) \mid z \in Z, x \in A, x \neq 1\},\ 0_K = (0,0), 1_K = (1,0)$$

$$(n,x) + (m,y) = \begin{cases} (n+m, x \oplus y) & \text{if } x \oplus y < 1 \\ (n+m+1, x * y) & \text{if } x \oplus y = 1 \end{cases}$$

$$-(n,x) = \begin{cases} (-n, 0) & \text{if } x = 0 \\ (-(n+1), \neg x) & \text{if } 0 < x < 1 \end{cases}$$

$$(n,x) \leq (m,y) \text{ if } n < m \text{ or } n = m \text{ and } x \leq y$$

$$(n,x) \times (m,y) = (nm, x \cdot y) + m(0,x) + n(0,y)$$

μŁΠ-algebras and ordered fields

- $\Phi(\mathcal{A})$ is a linearly ordered, commutative, integral domain
- Hence $\Phi(\mathcal{A})$ can be extended to a linearly ordered field by taking its fractions filed.
- The interval algebra of the resulting field is $\mathcal{A}$.
- Vice-versa, given a linearly ordered field $\mathcal{K}$ it is easy to construct a ŁΠ-algebra, $\Psi(\mathcal{K})$.

μŁΠ-algebras and ordered fields

Definition

Given a linearly ordered field $\mathcal{K}$ we define an ŁΠ-algebra $\mathcal{A}$, called the ŁΠ-interval algebra of $\mathcal{K}$, in the following way

$$A = \{x \in K \mid 0 \leq x \leq 1\}$$

$$x * y = \max(0, x + y - 1) \quad x \Rightarrow y = \min\{1, 1 - x + y\} \quad x \cdot y = x \times y$$

$$x \Rightarrow_\Pi y = \begin{cases} 1 & \text{if } x \leq y \\ z & \text{otherwise} \end{cases}$$

where z is the only element such that $y = x * z$

μŁΠ-algebras and real closed fields

Theorem

Any field associated to a μŁΠ-algebra is a real closed field.

Corollary

Every linearly ordered $\text{Ł}\Pi\frac{1}{2}$-algebra can be embedded in at most one linearly ordered μŁΠ-algebra (up to isomorphism).

Theorem

Every linearly ordered μŁΠ-algebra is isomorphic to the interval algebra of some real closed field. Conversely every real closed field is isomorphic to a real closed field associated to a linearly ordered μŁΠ-algebra.

μŁΠ-algebras and real closed f-semifields

- Such a correspondence can be extended to the whole class of μŁΠ-algebras. In order to do that the, rather technical, concept of real closed f-semifields has to be introduced.
- Real closed f-semifields can be seen as a lattice-ordered generalization of real closed fields, which are closed under subdirect products.

Theorem

There exists an equivalence of categories between the category of μŁΠ-algebras with their omomorphisms and the category of real closed f-semifields and their omomorphisms.

Some considerations

- In Łukasiewicz logic all the connectives have a continuous interpretation.
- Łukasiewicz logic is the only extension of BL whose connectives are all continuous.
- For this reason the study of Gödel and Product logic with fixed point in this setting can be carried out directly in BL.
- An algebraic perspective gave good results in the case of μŁΠ logic.

μMV$^-$-algebra

Definition

A μMV$^-$-algebra is a MV algebra, endowed with a function $\mu x_{t(x)}$ for any term $t(x)$ in the language of MV algebras, such that it satisfies the following conditions.

1. $\mu x_{t(x)} = t(\mu x_{t(x)})$
2. If $t(s) = s$ then $\mu x_{t(x)} \leq s$
3. If $\bigwedge_{i \leq n}(|p_i - q_i|) = 1$ then $\mu x.(t(p_1, \dots, p_n)) = \mu x.(t(q_1, \dots, q_n))$

Δ axioms

In ŁΠ-logic $\Delta(x)$ has been introduced as shorthand for $\neg(x) \to_{\Pi} 0$. Nevertheless the operator Δ can be also introduced axiomatically:

Definition

A *MV_{Δ}-algebra* is a MV algebra with an operator Δ that satisfies:

1. $\Delta(1) = 1$.
2. $\Delta(x \Rightarrow y) \leq \Delta(x) \Rightarrow \Delta(y)$.
3. $\Delta(x) \sqcup \neg\Delta(x) = 1$.
4. $\Delta(x) \leq x$.
5. $\Delta(\Delta(x)) = \Delta(x)$.
6. $\Delta(x \sqcup y) = \Delta(x) \sqcup \Delta(y)$.

Δ and linearly ordered algebras

In linearly ordered μMV$^-$-algebras the Δ operator can be simulated by μ-functions as

$$\Delta(y) = \neg\mu x_{(x\oplus\neg y)}(y)$$

Quite surprisingly, the fact that this term satisfies the axioms of Δ also in the non linear case is strictly tied to the subdirect representation of μMV$^-$-algebras.

Theorem

A μMV$^-$-algebra is the subdirect product of linearly ordered μMV$^-$-algebras iff the term $\neg\mu x_{(x\oplus\neg y)}(y)$ satisfies the axioms of Δ.

μMV-algebras

For the reason above, the μMV^{-}-algebras which satisfies the two equivalent conditions are called μMV-algebras.

Corollary

Every μMV algebra is the subdirect product of linearly ordered μMV algebras.

With Δ it is possible to express quasi-equations as equations, so this proves:

Corollary

The class of μMV algebras is a variety.

Divisible MV algebras

Definition

A divisible MV algebra is an MV algebra with a family of operators δ_n such that:

1. $(n)\delta_n(x) = x$
2. $\delta_n(x) \odot (n-1)\delta_n(x) = 0$

Where $(n)x$ is a shorthand for $\underbrace{x \oplus \ldots \oplus x}_{n-times}$

Definition

A divisible MV$_\Delta$algebra is a structure $\mathcal{A} = \langle A, \oplus, \neg, 0, \delta_n, \Delta\rangle$ such that:

- $\langle A, \oplus, \neg, 0, \Delta\rangle$ is a MV$_\Delta$algebra
- $\langle A, \oplus, \neg, 0, \delta_n\rangle$ is a divisible MV algebra

The operators δ_n

For any n, the operators δ_n, of divisible MV algebras, are definable by fixed points:

$$\delta_n(x) = \mu y_{(x \ominus (n-1)y)}(x)$$

Lemma

For every μMV algebra, the operator defined above satisfies:

1. $(n)\delta_n(x) = x$
2. $\delta_n(x) \odot (n-1)\delta_n(x) = 0$

Divisible MV$_\Delta$algebra

So we have reached the following result:

Theorem

Every μMV algebra contains a definable divisible MV$_\Delta$algebra.

In the following we will see what is needed to invert such a theorem

Pice-wise linear terms

- Every term of an MV algebra can be interpreted as a continuous pice-wise linear function with integer coefficents form $[0,1]^n$ to $[0,1]$.
- Parameterizing the function in all its variables but x this becomes of the form:

$$f(x) = \begin{cases} z_1 x \pm k_1 & \text{if } p_1 \leq x \leq q_1 \\ \vdots & \vdots \\ z_i x \pm k_i & \text{if } p_i \leq x \leq q_i \end{cases}$$

where $z_i \in \mathbb{Z}$ and k_i, p_i, q_i are polynomials in the variables parameterized.

Pice-wise linear terms

Such a function is the interpretation of a term of the form:

$$\bigvee_{i \leq I,} (\Delta(x \to q_i) \wedge \Delta(p_i \to x) \wedge c_i)$$

where c_i are the terms corresponding to $z_i x \pm k_i$

Term-wise equivalence

Theorem

μMV algebras and divisible MV_Δ algebras are term-wise equivalent.

First of all we find the minimum fixed points of some basic term. Let us define:

$$
\begin{array}{rcr}
\bar{\mu}x_c = \neg\Delta(\neg k) & \text{if} & c = (m)x \oplus k \\
\bar{\mu}x_c = \delta_{m-1}(k) & \text{if} & c = (m)x \ominus k \\
\bar{\mu}x_c = \delta_{m+1}(\neg k) & \text{if} & c = \neg((m)x \oplus k) \\
\bar{\mu}x_c = \delta_{m+1}(k) \oplus \delta_{m+1}(1) & \text{if} & c = \neg((m)x \ominus k)
\end{array}
$$

Where we have put $\delta_0(x) = 0$ for every x.
It is easy to see that in all four cases $\bar{\mu}x_c$ is the minimum fixed point of c.

Proof cont'd

To give the fixed point function associated to any term $t(x)$ we first find a term equivalent to $t(x)$ in which all the linear components are explicitly present, let it be

$$\bigwedge_{i \leq I}(\Delta(x \to q_i) \wedge \Delta(p_i \to x) \wedge c_i)$$

By continuity of the function a fixed point for this term must exist and it will be among the fixed point of the functions c_i.
So we define:

$$\mu x_{t(x)} = \bigwedge_{i \leq I}[\neg\Delta(t(\bar{\mu} x_{c_i}) \leftrightarrow \bar{\mu} x_{c_i}) \oplus \bar{\mu} x_{c_i}]$$

δ-lattice ordered groups

Once this equivalence is established it becomes fairly easy to extend known results (and techniques) about divisible MV algebras and MV$_\Delta$algebra to μMV algebras.

Definition

A δ-lattice ordered group (δ-ℓ-group, for short) is a structure $\mathcal{G} = \langle G, +, -, \wedge, \vee, \delta, 0, 1\rangle$ where $\langle G, +, -, \wedge, \vee, 0, 1\rangle$ is an abelian lattice ordered group and δ is a unary operation satisfying:

$$\delta(x) \leq |x| \wedge 1 \qquad \delta(\delta(x)) = \delta(x) \qquad \delta(x) = \delta(x \wedge 1)$$

$$\delta(1) = 1 \qquad \delta(x) \vee (1 - \delta(x)) = 1 \qquad 0 \leq \delta(x)$$

$$\delta(x) \wedge \delta(y^+ + (1 - |x|)^+) \leq \delta(y)$$

where $|x| = x \vee (-x)$ and $x^+ = x \vee 0$

μMV algebra and divisible δ-group.

Theorem (Montagna, 2001)

There is a functor Γ_Δ (extending Mundici's functor) between the category of MV$_\Delta$algebra and the category of δ-ℓ-groups which, together with its inverse, forms an equivalence of category.

Theorem

Each linearly ordered μMV algebra is isomorphic to the unitary interval of a linearly ordered divisible δ-ℓ-group.

Standard completeness

Theorem

The μMV algebra $\langle [0,1], \oplus, \neg, 0, 1, \mu x_{t(x)} \rangle$ generates the variety of μMV algebras.

Theorem

The free μMV algebra over a finite number of generator n is given by all the piecewise continuous and linear functions with rational coefficients form $[0,1]^n$ to $[0,1]$.

Amalgamation

Lemma (Montagna, 2006)

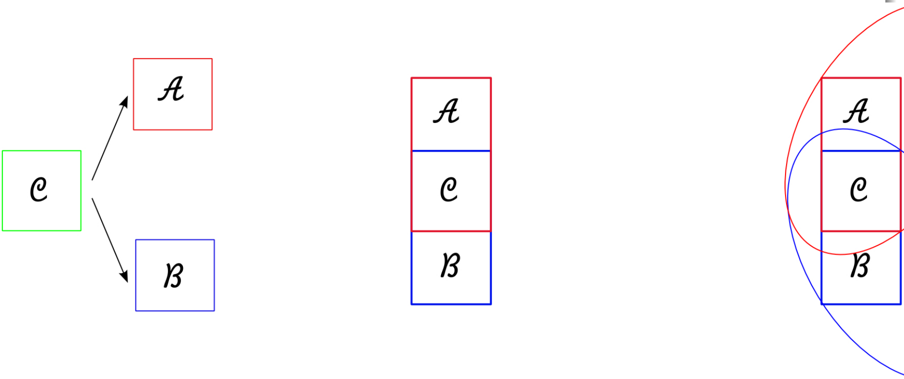
Let **K** *a quasi-variety of BL algebras possibly with additional operators such that* $\mathbf{K}_{lin}$ *has the amalgamation property. Then* **K** *has the amalgamation property.*

Theorem

Linearly ordered μMV *algebras enjoy amalgamation.*

Amalgamation

Proof.



□

Fixed points in BL-algebras

- The more general are the structure the harder is to understand how fixed point connectives behave.
- The operations which in general are continuous in a BL-algebra are only $*, \wedge$ and $\vee$
- Considering a minimum fixed point for continuous term makes little sense as for all y one has $\mu x_{x*y}(y) = \mu x_{x\vee y}(y), \mu x_{x\wedge y}(y) = 0$

νBL-algebras

Definition

A νBL-algebra is a structure

$$\mathcal{A} = \langle A, *, \Rightarrow, \vee, \wedge, 0, 1, \{\nu x_{t(x)}\}_{t(x) \in Cterm} \rangle$$

such that:

1. $\mathcal{A} = \langle A, *, \Rightarrow, \vee, \wedge, 0, 1 \rangle$ is a BL-algebra,
2. $t(\nu x_{t(x)}) = \nu x_{t(x)}$,
3. If $t(s) = s$ then $s \leq \nu x_{t(x)}$,
4. $\nu x_{x^2 * y}(y) \vee \nu x_{x^2 * z}(z) = \nu x_{x^2 * (y \vee z)}(y, z)$.

Subdirect representation νBL-algebras

Lemma

For every νBL-algebra and every $a \in \mathcal{A}$ the element given by $\nu x_{x^2 * y}(a)$ is the largest idempotent below a.

Lemma

Every subdirectly irreducible νBL is linearly ordered.

Theorem

νBL-algebras are exactly the subdirect products of linearly ordered νBL-algebras.

A normal form theorem for νBL-algebras

Lemma

Every term of the form $\nu x_{t(x)}$ is equivalent to

$$\bigwedge_{i \in I} \bigvee \nu x_{x^{n_i} * y}(a_i)$$

where, conventionally, we write $\nu x_{x^0 * y}(a) = a$. In other words in every continuous term the functions νx can be pushed inside until the basic parts of the form $x^n * a$.

Lemma

In every νBL-algebra $\nu x_{x^2 * y}(a) = \nu x_{x^n * y}(a)$ for every $n \geq 2$

Notable operators

Let us define $\Box(a) = \nu x_{x^2 * y}(a)$ and $\triangle(a) = \nu x_{x*y}(a)$.
Let us also use the following shorthands:

$$\triangle^n(a) = \triangle(\triangle^{n-1}(a)),$$
$$\triangle(a)^n = \underbrace{\triangle(a) * \ldots * \triangle(a)}_{n \text{ times}}.$$

General properties of $\Box$ and $\triangle$

Lemma

1. *For every* $n \geq 1$, $\triangle(a)^n \geq \Box(a)$,
2. *For every* $n \geq 1$, $\triangle^n(a) \geq \Box(a)$ *and* $\Box(\triangle^n(a)) = \Box(a)$,
3. *If* $m \leq n$ *then* $\triangle^m(a) \geq \triangle^n(a)$ *and* $\triangle(a)^m \geq \triangle(a)^n$,
4. *For any* $n \in \mathbb{N} \setminus \{0\}$ $\triangle(a)^n \geq \triangle^2(a)$.

Archimedean νBL-algebras

Theorem

Let $\mathcal{A}$ be a νBL-algebra, then for every $a \in \mathcal{A}$, the following are equivalent:

$$\exists n > 1(\triangle(a)^n = \triangle(a)),$$
$$\exists n > 1(\triangle^n(a) = \triangle(a)),$$
$$\triangle(a) = \Box(a).$$

Definition

An archimedean νBL-algebra is an algebra in which, for any element a, one of the equivalent conditions of theorem above holds.

Lemma

For any element a of a νBL-algebra the following hold:

1. $\exists n(\triangle^n(a) = \triangle^{n-1}(a))$ *if, and only if,* $\exists m(\triangle^m(a) = \Box(a))$;
2. $\exists n(\triangle(a)^n = \triangle(a)^{n-1})$ *if, and only if,* $\exists m(\triangle(a)^m = \Box(a))$;

Taxonomy of νBL-algebras

Definition

- A νBL-algebra is said $_\mathbf{m}$**-archimedean** if it satisfies $\triangle^m(a) = \Box(a)$.
- A νBL-algebra is said $^\mathbf{m}$**-archimedean** if it satisfies $\triangle(a)^m = \Box(a)$.
- A νBL-algebra is said $_\infty$**-archimedean** if it satisfies

$$\exists m > 1(\triangle^m(a) = \Box(a)).$$

- A νBL-algebra is said $^\infty$**-archimedean** if it satisfies

$$\exists m > 1(\triangle^m(a) = \Box(a)).$$

Taxonomy of νBL-algebras

- Notice that archimedean, $_m$-archimedean and ^{m}-archimedean νBL-algebras are equationally definable subclasses of νBL-algebras, hence sub-quasi-varieties.
- $_\infty$-archimedean and $^\infty$-archimedean can be seen as the unions, for $m \in \mathbb{N}$ of the quasi-varieties of $_m$-archimedean and ^{m}-archimedean, respectively.

Taxonomy of νBL-algebras

Figure 1 should make clear their reciprocal relationships.

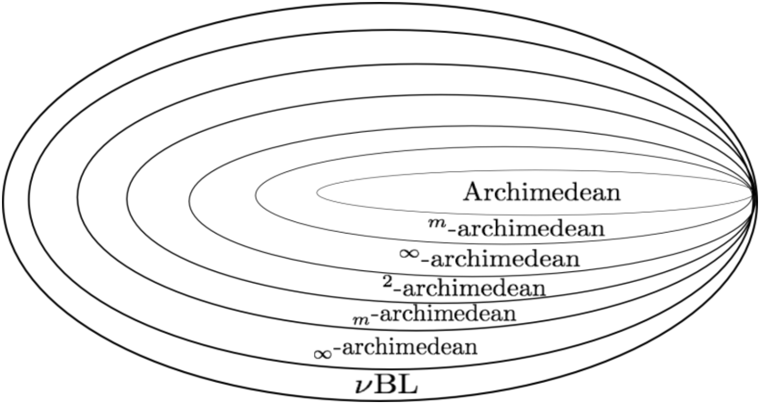


Figure: Subclasses of the variety of νBL-algebras

Further studies

- Explore more deeply the properties of μŁΠ-algebras. To which extent they can substitute real closed fields? Which properties of real closed fields have a meaningful interpretation in μŁΠ-algebras? Are there properties of μŁΠ-algebras which can give us new informations on real closed fields?
- Find other categorcial equivalences.
- We added all possible fixed points operations at once, but some of them proved to be remarkable operators. Characterize classes of fixed points as operators with particular properties.
- Explore the approach based on Kripke frames and find difference and analogies with the one given here.
- Study whether it is possible to use the same approach for first order many-valued logics.

Thank you for your attention.