Canonical formulas for k-potent residuated lattices

A joint work with N. Bezhanishvili and N. Galatos

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LATD
Vienna, 17th July 2014.

Main aim

1. We look for *uniform* axiomatisations of all substructural logics.
2. Possibly, with all axioms having a certain *shape*.
3. Possibly, with all axioms within a certain *complexity class*.
4. Possibly, with additional *nice properties* of the axioms.

We achieve this, for k-potent substructural logics, through *canonical formulas*.

Main result

Theorem

There exists a uniform way of axiomatising all k-potent extensions of FL_{ew} such that the axioms have a common shape, bounded complexity, and a semantic characterisation.

Caveats

1. Just in this talk, residuated lattice means commutative, integral, residuated lattice.
2. The short-hand k-RL will be often used to mean k-potent (commutative, integral) residuated lattice.
3. The methods will freely and informally move form algebra to logic and vice versa.

Canonical formulas (abstractly)

To every finite algebra A, associate a formula $\gamma(A, \dots)$ such that

1. $A \not\models \gamma(A, \dots)$.
2. There is a relation $\mathcal{R}$ such that for every Heyting algebra B we have

$$B \not\models \gamma(A, \dots) \quad \text{iff} \quad A \,\mathcal{R}\, B$$

Example: Jankov formulas

Theorem (Jankov, de Jongh$^{\delta}$)

For every finite, subdirectly irreducible Heyting algebra A there exists a formula $\chi(A)$ such that for every Heyting algebra B we have:

$$B \not\models \chi(A) \qquad \textit{iff} \qquad \exists C \quad A \rightarrowtail C \twoheadleftarrow B.$$

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For every finite, subdirectly irreducible Heyting algebra A there exists a formula $\chi(A)$ such that for every Heyting algebra B we have:

$$B \not\models \chi(A) \qquad \textit{iff} \qquad \exists C \quad A \rightarrowtail C \twoheadleftarrow B.$$

- Uncountably many intermediate logics can be axiomatised by these formulas.
- There are intermediate logics that are not axiomatisable by Jankov formulas.
- Every logic whose variety of Heyting algebras is locally finite can be axiomatised by Jankov formulas.

The case of intuitionistic logic

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- Fine (1974) and Rautenberg (1980) introduced modal logic analogues of these formulas.
- However, there exist intermediate and transitive modal logics that are not axiomatisable by Jankov or subframe formulas.
- Zakharyaschev (1988-92), refining Jankov and Fine's methods, introduced canonical formulas and showed that each intermediate and transitive modal logic is axiomatisable by canonical formulas.

The case of intuitionistic logic

- Zakharyaschev's method is model theoretic and his formulas have an extra parameter.

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- All intermediate logics can be axiomatised by those *algebraic* canonical formulas.
- Their method relies on locally finite reducts of Heyitng algebras.

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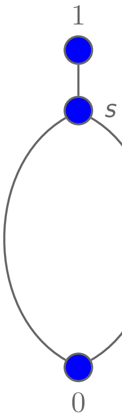
2. Special notion of morphisms: *D-maps*.
3. Characterisation of subdirectly irreducible algebras as the ones that have a second last element.

Subdirectly irreducible k-RL

Lemma

A k-potent residuated lattice is subdirectly irreducible if, and only if, it has a second last element.



Locally finite reducts

- Given a finite partial subalgebra B of a k-potent residuated lattice C, one can close the subalgebra for the operations $\vee$ and $\cdot$ still obtaining a finite algebra.

W. J. Blok and C. J. Van Alten. The finite embeddability property for residuated lattices, pocrims and BCK-algebras. Algebra Universalis, vol. 48(3), pp. 253–271, 2002.

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- Blok and Van Alten noticed that such an algebra can be endowed with $\wedge$ and $\to$, setting:

$$a \to b := \bigvee\{c \in B \mid a \cdot c \leq b\}$$
$$a \wedge b := \bigvee\{c \in B \mid c \leq a \text{ and } c \leq b\} .$$

  - the resulting structure A is a finite k-potent residuated lattice.
  - the newly defined operations $\wedge$, $\to$ agree with the partial ones already defined on B.

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D-maps

Recall from the previous slide

B finite partial subalgebra of C,
A finite algebra generated as above.

If $D^{\wedge}, D^{\rightarrow} \subseteq B \subseteq A$ are the subsets of pairs of elements on which $\wedge$ and $\rightarrow$ are defined, then the embedding of A into C preserves:

1. $\vee$ and $\cdot$ for all elements of A,
2. $\wedge$ only for the elements of $D^{\wedge}$,
3. $\rightarrow$ only for the elements of $D^{\rightarrow}$.

D-maps

The previous slide motivates the following definition:

Definition

Let $D := (D^{\wedge}, D^{\to})$ be any pair of subsets of A^2. A D-map between residuated lattices A and B is an injective function $f\colon A \to B$ such that for all $a, b \in A$

1. $f(a \vee b) = f(a) \vee f(b)$ and $f(a \cdot b) = f(a) \cdot f(b)$,
2. If $(a, b) \in D^{\wedge}$ then $f(a \wedge b) = f(a) \wedge f(b)$,
3. If $(a, b) \in D^{\to}$ then $f(a \to b) = f(a) \to f(b)$,

Notation

$$A \overset{D}{\rightarrowtail} B.$$

Canonical formulas

Let $(A, D^{\wedge}, D^{\rightarrow})$ be a triplet such that A is a finite k-potent residuated lattice and $D^{\wedge}, D^{\rightarrow} \subseteq A^2$.

$$
\begin{aligned}
\Gamma := \quad & (X_0 \leftrightarrow \bot) \wedge (X_1 \leftrightarrow \top) \wedge \\
& \bigwedge \{X_{a \cdot b} \leftrightarrow X_a \cdot X_b \mid a, b \in A\} \wedge \\
& \bigwedge \{X_{a \vee b} \leftrightarrow X_a \vee X_b \mid a, b \in A\} \wedge \\
& \bigwedge \{X_{a \rightarrow b} \leftrightarrow X_a \rightarrow X_b \mid (a, b) \in D^{\rightarrow}\} \\
& \bigwedge \{X_{a \wedge b} \leftrightarrow X_a \wedge X_b \mid (a, b) \in D^{\wedge}\} \\
\Delta := \quad & \bigvee \{X_a \rightarrow X_b \mid a, b \in A \text{ with } a \not\leq b\} .
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Where we introduced a fresh variable X_a for each $a \in A$.

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\end{aligned}
$$

Where we introduced a fresh variable X_a for each $a \in A$. Define the *canonical formula* $\gamma(A, D^{\wedge}, D^{\rightarrow})$ associated to A, $D^{\wedge}$, and $D^{\rightarrow}$ as:

$$\gamma(A, D^{\wedge}, D^{\rightarrow}) := \Gamma^k \rightarrow \Delta \ .$$

Canonical formulas [Cont'd]

- Define the *canonical evaluation* ν to be the assignment that sends $X_a \mapsto a$.
- Notice that if A in addition is subdirectly irreducible, then
  - $\nu(\Gamma^k) = 1$,
  - $\nu(\Delta) = s$ where s is the coatom of A,
  - in particular $A \not\models \gamma(A, D^{\wedge}, D^{\rightarrow})$.

Proof of the Claim

CLAIM

$B \not\models \gamma(A, D^{\wedge}, D^{\to})$ if, and only if, $A \overset{D}{\rightarrowtail} C \twoheadleftarrow B$.

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$B \not\models \gamma(A, D^{\wedge}, D^{\to})$ if, and only if, $A \rightarrowtail \!\!\!\!\!\!\! D \rightarrow C \twoheadleftarrow B$.

$\Leftarrow$

- The assignment $\mu(X_a) := h(\nu(X_a)) = h(a)$ falsifies γ in C (so γ also fails in B, as C is its epimorphic image).
- Indeed, $\mu(\Gamma^k) = 1$, because h preserves exactly what needed.
- $\mu(\Delta) = s$, for h is injective, hence $a \not\leq b$ implies $h(a) \not\leq h(b)$. So $\mu(X_a \to X_b) = h(a) \to h(b) \neq 1$.

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$\Rightarrow$

- Let v into B such that $v(\Gamma^k) \not\leq v(\Delta)$.
- Take a s.i. quotient $p \colon B \twoheadrightarrow C$ such that $p \circ v(\Gamma^k) = 1$ and $p \circ v(\Delta) \neq 1$
- Define $h \colon A \to C$ by $h(a) := p \circ v(X_a)$ and show that h is a D-map.

Main theorem

Theorem

Whenever $FL_{ew}^{k} \nvdash \varphi$*, there exist triplets* $(A_1, D_1^{\wedge}, D_1^{\rightarrow}), \ldots, (A_m, D_m^{\wedge}, D_m^{\rightarrow})$ *such that for any subdirectly irreducible* B*,*

$$B \models \varphi \quad \text{if, and only if,} \quad B \models \bigwedge_{i=1}^{m} \gamma(A_i, D_i^{\wedge}, D_i^{\rightarrow}).$$

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Whenever $FL^k_{ew} \nvdash \varphi$, *there exist triplets* $(A_1, D^{\wedge}_1, D^{\rightarrow}_1), \ldots, (A_m, D^{\wedge}_m, D^{\rightarrow}_m)$ *such that for any subdirectly irreducible* B,

$B \models \varphi$ *if, and only if,* $B \models \bigwedge_{i=1}^{m} \gamma(A_i, D^{\wedge}_i, D^{\rightarrow}_i)$.

$\star_1$: $B \models \varphi \;\leftrightarrow\; A_i \overset{D}{\rightarrowtail} C \twoheadleftarrow B$

$\star_2$: $A_i \overset{D}{\rightarrowtail} C \twoheadleftarrow B \;\leftrightarrow\; B \models \bigwedge_{i=1}^{m} \gamma(A_i, D^{\wedge}_i, D^{\rightarrow}_i)$

The main steps in the proof

Step 1

Formula

φ axiomatises a proper extension of FL^k_{ew},

$\rightsquigarrow$

Finite algebras

Triplets
$(A_1, D_1^{\wedge}, D_1^{\rightarrow}), \ldots$
$(A_m, D_m^{\wedge}, D_m^{\rightarrow})$

Step 2

Finite algebras

Triplets
$(A_1, D_1^{\wedge}, D_1^{\rightarrow}), \ldots$
$(A_m, D_m^{\wedge}, D_m^{\rightarrow})$

$\rightsquigarrow$

Formula

Canonical formula
$\gamma(A, D_i^{\wedge}, D_i^{\rightarrow})$

Main steps [Cont'd]

Step 2 simply amounts to consider the canonical formulas of the triplets.

Step 2

Finite algebras

Triplets
$(A_1, D_1^{\wedge}, D_1^{\rightarrow}), \ldots$
$(A_m, D_m^{\wedge}, D_m^{\rightarrow})$

$\rightsquigarrow$

Formula

Canonical formula
$\gamma(A, D_i^{\wedge}, D_i^{\rightarrow})$

As seen in the CLAIM the formulas $\gamma(A, D_i^{\wedge}, D_i^{\rightarrow})$ are such that:

$$\exists i \quad B \not\models \gamma(A_i, D_i^{\wedge}, D_i^{\rightarrow}) \iff \exists i \exists C \quad A_i \rightarrowtail\!\!\!\!\!\not D\!\!\rightarrow C \twoheadleftarrow B \quad (\star_1)$$

Main steps [Cont'd]

Step 1

- If $\varphi(X_1, \dots, X_n)$ is not provable in FL_{ew}^k then it fails in the free n-generated k-RL $\mathscr{F}_n$.
- Consider the partial subalgebra $\mathsf{Sub}(\varphi)$ of $\mathscr{F}_n$ given by all subformulas of $\varphi(X_1, \dots, X_n)$.
- Apply Block&van Alten's construction to $\mathsf{Sub}(\varphi)$ and obtain a finite k-RL A.
- Consider the sets:

$$D^{\wedge} := \{(\psi_1, \psi_2) \mid \psi_1 \wedge \psi_2 \in \mathsf{Sub}(\varphi)\}$$
$$D^{\rightarrow} := \{(\psi_1, \psi_2) \mid \psi_1 \rightarrow \psi_2 \in \mathsf{Sub}(\varphi)\}.$$

- Take all subdirectly irreducible epimorphic images of A that refute φ, $p_i \colon A \to A_i$ with $i \leq m$.
- Consider the triplets $(A_i, D_i^{\wedge}, D_i^{\rightarrow})$ where $D_i^{\wedge} := p_i[D^{\wedge}]$ and $D_i^{\rightarrow} := p_i[D^{\rightarrow}]$.

The property $\star_2$ (right-to-left)

The property $\star_2$

$$B \not\models \varphi \quad \Longleftrightarrow \quad \exists i \leq m \exists C \quad A_i \rightarrowtail\!\!\!\!\!\rightarrow C \twoheadleftarrow B$$

$\boxed{\Leftarrow}$

- Fix a non-derivable φ and suppose that the associated triplets are such that $h\colon A_i \rightarrowtail\!\!\!\!\!\rightarrow C \twoheadleftarrow B$.
- If $a \to b \in \mathrm{Sub}_{A_i}(\varphi)$, then $h(a \to b) = h(a) \to h(b)$, and if $a \wedge b \in \mathrm{Sub}_{A_i}(\varphi)$, then $h(a \wedge b) = h(a) \wedge h(b)$.
- But $\varphi(h_i(X_1), \ldots, h_i(X_n)) \neq 1$ in A_i, so $\varphi(h(h_i(X_1)), \ldots, h(h_i(X_n))) \neq 1$ in C.
- Finally, φ fails also in B, as C is a homomorphic image of B.

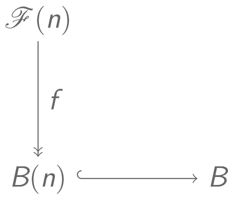
The property $\star_2$ (left-to-right)

The property $\star_2$

$$B \not\models \varphi \quad \iff \quad \exists i \leq m \exists C \quad A_i \rightarrowtail D \rightarrow C \twoheadleftarrow B$$

$\Rightarrow$

- Suppose that $B \models \varphi(b_1, ..., b_n) \neq 1$.
- Let $B(n)$ be the algebra generated by $b_1, ..., b_n$.
- Let $f\colon \mathscr{F}(n) \twoheadrightarrow B(n)$ extend the assignment $X_1 \mapsto b_1, \ldots, X_n \mapsto b_n$.

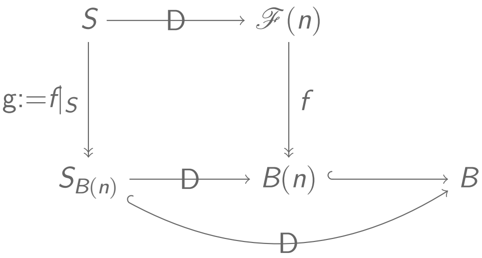


The property $\star_2$ (left-to-right)

The property $\star_2$

$$B \not\models \varphi \iff \exists i \leq m \, \exists C \quad A_i \overset{D}{\rightarrowtail} C \twoheadleftarrow B$$

- $\mathrm{Sub}(\varphi) :=$ set of subpolynomials of φ in $\mathscr{F}(n)$ and $\mathrm{Sub}_{B(n)}(\varphi) := f[\mathrm{Sub}(\varphi)]$.
- $S := (\cdot, \vee)$-subalgebra of $\mathscr{F}(n)$ generated by $\mathrm{Sub}(\varphi)$,
- $S_{B(n)} := (\cdot, \vee)$-subalgebra of $B(n)$ gen'd by $\mathrm{Sub}_{B(n)}(\varphi)$.
- $S_{B(n)}$ is subdirectly irreducible, so it is equal to some A_i.



Further research

- Drop integrality.
- Substitute commutativity with some weaker form like $xyx = xxy$.
- Try to replace locally finite with Finite Embeddability Property.
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- Study particular cases of canonical formulas.

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Thank you!