

Free $\mu\text{L}\Pi$ algebras

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T-norm based logics

Definition

A **t-norm** $*$ is a function from $[0, 1]^2$ to $[0, 1]$ which is:

- associative: $x(y * z) = (x * y) * z$,
- commutative: $x * y = y * x$,
- non-decreasing: $x \leq y$ implies $x * z \leq y * z$,
- $x * 1 = x$ and $x * 0 = 0$,

Definition

A **residual** \rightarrow , of a t-norm $*$, is a function from $[0, 1]^2$ to $[0, 1]$ such that

$$x * y \leq z \text{ if, and only if, } x \leq y \rightarrow z$$

T-norm based logics

Definition

Gödel logic is the logic complete w.r.t the following connectives:

$$x \wedge y = \min\{x, y\} \quad x \rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Product logic is the logic complete w.r.t. the following connectives:

$$x \cdot y = xy \quad x \rightarrow_{\Pi} y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$$

Łukasiewicz logic is the logic complete w.r.t:

$$x \oplus y = \min\{x + y, 1\} \quad \neg x = 1 - x$$

Mostert and Shields' Theorem

The tree above logics are the most important logics based on continuous t-norm because of the following result:

Theorem (Mostert, Shields '57)

Every continuous t-norm is locally isomorphic to either Łukasiewicz, product or Gödel t-norm.

Such a **decomposition** applies also for the algebraic semantic of continuous t-norm based logic.

Algebraic semantics

Theorem (Chang '58)

The algebraic semantic of Łukasiewicz logic is given by MV-algebras.

Theorem (Hájek '98)

The algebraic semantic of product logic is given by Π -algebras.

$\text{Ł}\Pi$ logic

Definition

The logic $\text{Ł}\Pi$ is the logic having as set of primitive connectives $\{\oplus, \neg, \cdot, \rightarrow_{\Pi}, 0, 1\}$ satisfying the following axioms:

- all axioms of Łukasiewicz logic for $\{\oplus, \neg, 0, 1\}$,
- all axioms of product logic for $\{\cdot, \rightarrow_{\Pi}, 0, 1\}$,
- $\varphi \cdot (\psi \ominus \xi) \leftrightarrow (\varphi \cdot \psi) \ominus (\varphi \cdot \xi)$,
- $\Delta(\varphi \rightarrow \psi) \rightarrow \varphi \rightarrow_{\Pi} \psi$.

Where $\Delta(\varphi)$ is defined as $(\neg\varphi) \rightarrow_{\Pi} 0$.

The rules are modus ponens and necessitation:

- If φ and $\varphi \rightarrow \psi$ then ψ ,
- if φ then $\Delta(\varphi)$.

Importance of $\text{Ł}\Pi$ logic

$\text{Ł}\Pi$ logic has a stronger expressive power than the above mentioned logics, indeed:

Theorem (Esteva, Godo, Montagna '01)

$\text{Ł}\Pi$ logic faithfully interprets Łukasiewicz, product and Gödel logic. Moreover, if limited to finite deduction, also Pavelka logic is interpretable in $\text{Ł}\Pi$ logic.

More generally:

Theorem (Marchioni, Montagna '06)

Every logic based on a continuous t -norm with a finite number of idempotents is definable in $\text{Ł}\Pi$ logic.

$\text{L}\Pi$ algebras

Definition

$\text{L}\Pi$ algebras are the algebraic semantic of $\text{L}\Pi$ logic, so they are structures of type $\mathcal{A} = \langle A, \oplus, \neg, \cdot, \rightarrow_{\Pi}, 0, 1 \rangle$.

Example

The algebra $\langle [0, 1], \oplus, \neg, \cdot, \rightarrow_{\Pi}, 0, 1 \rangle$, where the operations are defined as follows:

- $x \oplus y = \min\{x + y, 1\}$ $\neg x = 1 - x$
- $x \cdot y = xy$ (ordinary product between reals)
- $x \rightarrow_{\Pi} y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$

is a $\text{L}\Pi$ -algebra. Moreover it generates the variety of $\text{L}\Pi$ -algebras.

$\mu\text{L}\Pi$ logic

We introduce now the main subject: namely the logic $\text{L}\Pi$ with fixed points.

Definition

The logic $\mu\text{L}\Pi$ is obtained as an expansion of $\text{L}\Pi$ logic with a new (generalized) connective

$$\mu_{\varphi(p)}(\bar{\psi})$$

for any $\text{L}\Pi$ -formula $\varphi(p, \bar{\psi})$ in which the symbol \rightarrow_{Π} does not appear.

Such connectives must satisfy a number of axioms which we give directly in their algebraic form.

$\mu\text{L}\Pi$ algebras

Definition

Let us call $C\text{Term}$ the set of $\text{L}\Pi$ terms in which the symbol \rightarrow_{Π} does not appear. $\mu\text{L}\Pi$ -algebras are structures of type

$$\mathcal{A} = \langle A, \oplus, \neg, \cdot, \rightarrow_{\Pi}, 0, 1, \{\mu X_{t(x, \bar{y})}\}_{t(x, \bar{y}) \in C\text{Term}} \rangle$$

which satisfy the following axioms:

$\langle A, \oplus, \neg, \cdot, \rightarrow_{\Pi}, 0, 1 \rangle$ is a $\text{L}\Pi$ -algebra

$$\mu X_{t(x)}(\bar{y}) = t(\mu X_{t(x)}(\bar{y}), (\bar{y})),$$

If $t(s(\bar{y}), \bar{y}) = s(\bar{y})$ then $\mu X_{t(x)}(\bar{y}) \leq s(\bar{y})$,

$$\bigwedge_{i \leq n} \Delta(p_i \leftrightarrow q_i) \leq (\mu X_{t(x, \bar{y})}(p_1, \dots, p_n) \leftrightarrow \mu X_{t(x, \bar{y})}(q_1, \dots, q_n))$$

The $\mu\text{L}\Pi$ algebra on $[0,1]$

Example

The algebra $\langle [0, 1], \oplus, \neg, \cdot, \rightarrow, \perp, 0, 1, \{\mu X_{t(x)}(\bar{y})\}_{t(x, \bar{y}) \in CTerm} \rangle$ is a $\mu\text{L}\Pi$ -algebra.

Theorem

The $\mu\text{L}\Pi$ -algebra on $[0, 1]$ defined above generates the variety of $\mu\text{L}\Pi$ -algebras.

Some notation

Notation

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}^{\text{alg}}$ are, respectively, the sets of integer, rational and real algebraic numbers.
- $\mathbb{Z}[x_1, \dots, x_n]$ is the domain of polynomials in n variables and integer coefficients.
- $\mathbb{Q}(x_1, \dots, x_n)$ is its fraction field.
- We use the same symbols to denote their members and the associated functions from \mathbb{R}^n to \mathbb{R} .
- We will write $\{P > 0\}$ for $\{v \in [0, 1]^n \mid P(v) > 0\}$.

A remark

Remark

Notice that there is a tight link between fixed points and roots of polynomials. Indeed given a polynomial $P(x)$ its set of solutions is the set of fixed points of the polynomial $P(x) - x$. Viceversa, its set of fixed points can be seen as the set of solutions of the polynomial $P(x) + x$. This correspondence is preserved also when we restrict to the $[0, 1]$ interval.

Super-algebraic functions

Definition

Let us fix any $n \in \mathbb{N}$. We will call **root function** every function $f(y_1, \dots, y_n)$ such that for any $P(x) = \sum_{i \leq n} a_i x^i \in \mathbb{R}^{\text{alg}}[x]$:

$$f(a_1, \dots, a_n) = r \text{ iff } r \text{ is the minimum value such that } P(r) = 0.$$

We will call **super-algebraic** every function which is

- a rational polynomial function $P/Q \in \mathbb{Q}(x_1, \dots, x_n)$ or,
- a root function, or
- a composition of the previous two kinds of function.

Super-algebraic functions

Note that root functions are not enough for our description since an element of $t \in CTerm$ can be represented as a member of $\mathbb{Z}[x_1, \dots, x_n]$. Hence we need a different root function for every function represented by an element of $\mathbb{Z}[x_1, \dots, x_n]$.

Definition

Let \mathcal{R} be the set of all functions f_R such that given $f \in \mathbb{Z}[x, x_1, \dots, x_n]$

$$f_R[a_1, \dots, a_n] = r$$

iff

r is the minimum value for which $f(r, a_1, \dots, a_n) = 0$.

Super-algebraic functions

Lemma

\mathcal{R} is the set of super-algebraic functions.

Proof.

Notice that a rational polynomial function $P/Q \in \mathbb{Q}(x_1, \dots, x_n)$ is the function in \mathcal{R} associated to the polynomial $P(x_1, \dots, x_n) - xQ(x_1, \dots, x_n)$.

For the other direction notice that composing a root function with a suitable projection gives any desired function in \mathcal{R} . \square

Galois' theorem

Theorem (Galois 1812)

A polynomial is solvable by radicals if, and only if, the group of automorphisms of the field of its solutions which fix the field of coefficients is solvable.

In particular there exist polynomials which are not solvable by radicals. Hence super-algebraic functions are not only algebraic functions.

What we have to add to algebraic functions to get super-algebraic functions is **unknown**.

Free algebras

Notation

The free $\mu\text{L}\Pi$ -algebra over κ generators will be denoted by $\mathcal{F}_\kappa(\mu\text{L}\Pi)$.

$\mathcal{F}_\kappa(\mu\text{L}\Pi)$ is the subalgebra of the algebra of all functions from $[0, 1]^\kappa$ to $[0, 1]$ generated by the projections closed under $\mu\text{L}\Pi$ operations defined point-wise.

In the case of MV-algebras, the characterization comes from a classical result:

Theorem (McNaughton '51)

The functions generated by the projections under MV-operations are exactly the continuous piecewise linear functions with integer coefficients.

$\mathcal{F}_0(\mu\text{L}\Pi)$

We will start with $\mathcal{F}_0(\mu\text{L}\Pi)$. Such an algebra is isomorphic to the interval algebra of \mathbb{R}^{alg} . Indeed something stronger holds:

Proposition

The $\mu\text{L}\Pi$ -algebra on the $[0, 1]$ interval of \mathbb{R}^{alg} can be embedded in every linearly ordered $\mu\text{L}\Pi$ -algebra.

Semialgebraic sets

Definition

A subset S of $[0, 1]^n$ is a **\mathbb{Q} -semialgebraic** if it is a boolean combination of sets of the form $\{P > 0\}$ for some $P \in \mathbb{Z}[x_1, \dots, x_n]$.

A subset S of $[0, 1]^n$ is a **semialgebraic** if it is a boolean combination of sets of the form $\{P > 0\}$ for some $P \in \mathbb{R}^{alg}[x_1, \dots, x_n]$.

Hats

Definition

A $\mathbb{L}\Pi$ -*hat* is a function $h : [0, 1]^n \longrightarrow [0, 1]$ such that there exists a \mathbb{Q} -semialgebraic set S and a function $f = P/Q \in \mathbb{Q}(x_1, \dots, x_n)$ such that:

- Q has no zero in S ,
- if $x \in S$ then $h(x) = f(x)$,
- if $x \notin S$ then $h(x) = 0$.

A μ -*hat* is a function $h : [0, 1]^n \longrightarrow [0, 1]$ such that there exists a semialgebraic set S and a super-algebraic function f such that if $x \in S$ then $h(x) = f(x)$ and if $x \notin S$ then $h(x) = 0$.

If h is a function which satisfies either of these conditions we will indicate it by $\langle S, f \rangle$.

Basic functions

Definition

A basic $\mathbb{L}\Pi$ -function and a basic μ -function over $[0, 1]^n$ are, respectively, a finite sum of $\mathbb{L}\Pi$ -hats and a finite sum of μ -hats

$$\langle S_1, f_1 \rangle + \langle S_2, f_2 \rangle + \dots + \langle S_k, f_k \rangle$$

such that $S_i \cap S_j = \emptyset$ for any $i \neq j$.

We will denote by $\mathbb{L}\Pi B_n$ and B_n , respectively, the sets of $\mathbb{L}\Pi$ -basic functions over $[0, 1]^n$ and μ -basic functions over $[0, 1]^n$

The algebra B_n

Theorem (Montagna, Panti '01)

$\mathcal{L}\Pi B_n$ is the free $\mathcal{L}\Pi$ -algebra over n generators.

Lemma

B_n contains the projection functions and is a $\mu\mathcal{L}\Pi$ -algebra under point-wise operations.

Proof.

An easy adaptation of the proof of the previous theorem shows that B_n is closed under $\mathcal{L}\Pi$ operations. Given a term in $t \in Cterm$ let

$$g = \langle S_1, P_1 \rangle + \dots + \langle S_r, P_r \rangle$$

be its associated function. □

The algebra B_n

Proof cont'd.

Then we claim

$$\mu^{X_t(x, \bar{y})} = \langle T_1, Q_1 \rangle + \dots + \langle T_r, Q_r \rangle$$

where T_i are semialgebraic sets and Q_i are μ -hat. Indeed if we associate to any Q_i a new polynomial $Q'_i = Q_i - x$ and call $R_{Q'_i}$ the functions which give the minimum root of the polynomial Q'_i , then it is easy to check that:

$$\begin{aligned} \mu^{X_t(x, \bar{y})} = & \langle \{\bar{y} \mid \exists z (Q_1(z, \bar{y}) = z \wedge (z, \bar{y}) \in T_1)\}, R_{Q'_1} \rangle + \\ & \vdots \\ & + \langle \{\bar{y} \mid \exists z (Q_r(z, \bar{y}) = z \wedge (z, \bar{y}) \in T_r)\}, R_{Q'_r} \rangle \end{aligned}$$



The algebra B_n

Proof cont'd.

But for every $1 \leq i \leq r$ the set

$$\{\bar{y} \mid \exists z (Q_i(x, \bar{y}) = x \wedge (x, \bar{y}) \in T_i)\}$$

is a projection of a semialgebraic set.

Hence by Tarski-Seidenberg theorem it is in turn a semialgebraic set. Moreover they are all disjoint, since the sets T_i are disjoint, and $R_{Q_i}(\bar{y}) \neq 0$ as $(x, \bar{y}) \in T_i$.

From this follows that if f_1, \dots, f_n are functions in B_n then $\mu_{X_t(x)}(f_1, \dots, f_n)$ is also in B_n . □

Lemma

Let $P \in \mathbb{R}^{alg}[x_1, \dots, x_n]$ and let $P^\sharp : [0, 1]^n \longrightarrow [0, 1]$ be defined for any $\vec{v} \in [0, 1]^n$ by $P^\sharp(\vec{v}) = \min\{\max\{P(\vec{v}), 0\}, 1\}$. The $P^\sharp \in \mathcal{F}_\kappa(\mu\text{L}\Pi)$.

Corollary

The characteristic function of every semialgebraic set is in $\mathcal{F}_\kappa(\mu\text{L}\Pi)$

Proof.

Since $\mathcal{F}_\kappa(\mu\text{L}\Pi)$ is closed under the boolean operator it suffices to prove that the characteristic functions of the sets of the form $\{P > 0\}$ are in $\mathcal{F}_\kappa(\mu\text{L}\Pi)$. But such a function is just $\neg\Delta(P^\sharp)$ \square

$\mathcal{F}_\kappa(\mu\text{L}\Pi)$

Theorem

$\mathcal{F}_\kappa(\mu\text{L}\Pi)$ is the algebra of piecewise super-algebraic functions from $[0, 1]^\kappa$ to $[0, 1]$

Proof.

We need to show that every basic function is in $\mathcal{F}_\kappa(\mu\text{L}\Pi)$. Since the semialgebraic sets appearing in the definition of a basic function are pairwise disjoint we can substitute every $+$ with \oplus . Hence it is sufficient to show that every μ -hat is in $\mathcal{F}_\kappa(\mu\text{L}\Pi)$, which easily comes from the definition. \square

The forgetful functor

Definition

Let us call \mathcal{F} the forgetful functor from the category of $\mu\mathbb{L}\Pi$ -algebras with their morphisms to the category of $\mathbb{L}\Pi$ -algebras with their morphisms.

For general reasons this functor has an adjoint \mathcal{G} which, given a $\mathbb{L}\Pi$ -algebra \mathcal{A} , gives the free $\mu\mathbb{L}\Pi$ -algebra on \mathcal{A} . Since the new structure is based on old terms, this construction has a number of nice properties.

The forgetful functor

Proposition

The functor \mathcal{G} creates isomorphisms.

Proposition

Given two linearly ordered $\mu\mathbb{L}\Pi$ -algebras they are isomorphic if, and only if, their underlying $\mathbb{L}\Pi$ -algebras are isomorphic.