Denominator respecting maps

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Embedding spaces

It is well known that

every compact Hausdorff space X can be embedded in some hypercube $[0,1]^J$ for some index set J.

Suppose that X is now endowed with a function $\delta\colon X \to \mathbb{N}$.

Problem

Given a pair $\langle X, \delta \rangle$, is there a continuous embedding $\iota\colon X \to [0,1]^J$ in such a way that the denominators of the points in $\iota[X]$ agree with δ?

Let us assume that "agree" means that $\delta(x) = \mathtt{den}(\iota(x))$.

Denominators

Recall that $\mathbb{N}$ forms a complete lattice under the divisibility order: the top being 0 and the bottom being 1.

Let J be a set and $\overline{p} \in [0,1]^J$. If $\overline{p} \in \mathbb{Q}^J$ we define its denominator to be the natural number

$$\mathtt{den}(\overline{p}) = \mathtt{lcd}\{p_i \mid i \in J\}$$

where $\mathtt{lcd}$ stands for the least common denominator. If $\overline{p} \notin \mathbb{Q}^J$ we set $\mathtt{den}(\overline{p}) = 0$.

1. A function $f\colon [0,1]^J \to [0,1]$ preserves denominators if for any $\overline{x} \in [0,1]^J$, $\mathtt{den}(f(x)) = \mathtt{den}(x)$.
2. A function $f\colon [0,1]^J \to [0,1]$ respects denominators if for any $\overline{x} \in [0,1]^J$, $\mathtt{den}(f(x)) \mid \mathtt{den}(x)$.

An easy counter-example

Consider $X = [0, 1]$ with its Euclidean topology and endow it with a constant δ:

$$\forall x \in X \quad \delta(x) = 1.$$

The only points with denominator equal 1 in $[0, 1]^J$ are the so-called lattice points i.e., points whose coordinates are either 0 or 1.

The only way ι could agree with δ is to send all points in one lattice point —failing injectivity— or by sending the points in different lattice points —failing continuity.

MV-algebras

The above mentioned problem is crucial in the duality theory of MV-algebras —the equivalent algebraic semantics of Łukasiewicz logic.

An MV-algebra is a structure $\langle A, \oplus, \neg, 0\rangle$ such that

1. $\langle A, \oplus, 0\rangle$ is a commutative monoid,
2. $\neg\neg x = x$,
3. $\neg 0 \oplus x = \neg 0$
4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Example

The interval $[0, 1]$ in the real numbers has a natural MV-structure given by the truncated sum $x \oplus y = \mathtt{min}\{x + y, 1\}$ and $\neg x = 1 - x$. The importance of this structure comes from the fact that it generates the whole variety of MV-algebras.

MV-algebras and compact spaces

Theorem (Marra, S. 2012)

Semisimple MV-algebras with their homomorphisms form a category that is dually equivalent to the category of compact Hausdorff spaces embedded in some hypercube, with $\mathbb{Z}$-maps among them.

Definition

For I, J arbitrary sets, a map from $\mathbb{R}^I$ into $\mathbb{R}^J$ is called $\mathbb{Z}$-map if it is continuous and piecewise (affine) linear map, where each (affine) linear piece has integer coefficients.

Remark

Since every $\mathbb{Z}$-map f acts on each point as an linear function with integer coefficients, it respect denominators i.e.,

$$\mathtt{den}(f(x)) \mid \mathtt{den}(x).$$

Mundici's functor

An abelian ℓ-group with order unit (uℓ-group, for short), is a partially ordered Abelian group G whose order is a lattice, and that possesses an element u such that

$$\text{for all } g \in G, \text{ there exists } n \in \mathbb{N} \text{ such that } (n)u \geq g.$$

The functor Γ that takes an uℓ-group $\langle G, u\rangle$ to its unital interval $[0, u]$ with operation $\oplus$ and $\neg$ defined as follows:

$$x \oplus y = \mathtt{min}\{u, x + y\} \qquad \text{and} \qquad \neg x = u - x,$$

is full, faithful, and dense hence it has a quasi-inverse Ξ and

Theorem (Mundici 1986)

The pair Γ, Ξ gives an equivalence of categories between the category of MV-algebras with their morphisms, and the category of uℓ-groups with ordered group morphisms preserving the order unit.

Norm induced by the order unit

Definition

Let (G, u) be a $u\ell$-group. The order unit u induces a seminorm $\| \ \|_u$ defined as folows:

$$\|g\|_u := \inf \left\{ \frac{p}{q} \in \mathbb{Q} \mid p, q \in \mathbb{N}, q \neq 0 \text{ and } q|g| \leq pu \right\}$$

The seminorm $\| \ \|_u \colon G \to R^+$ is in fact a norm if, and only if, G is archimedean. Any semisimple MV-algebra A inherits a norm from its enveloping (archimedean) group $\Xi(A)$.

Definition

An norm-complete MV-algebra is a semisimple MV-algebra which is Cauchy-complete w.r.t. its induced norm.

Stone-Yosida-Kakutani duality

Theorem (Stone-Yosida-Kakutani duality 1941)

A unital real vector space (V, u) is isomorphic to $(\mathrm{C}(X), 1)$ for some compact Hausdorff space X, if, and only if, V is archimedean and norm-complete (with respect to the norm $\| \|_u$ induced by the unit).

Question

What if we want to substitute $u\ell$-group for real vector space in the above statements?

Remark

An answer was already given by Stone: compact Hausdorff spaces correspond to archimedean, complete and divisible $u\ell$-groups.

Denominator preserving maps

Theorem (Goodearl-Handelman 1980)

Let X be a compact Hausdorff space. For any $x \in X$ choose A_x to be either $A_x = \mathbb{R}$ or $A_x = (\frac{1}{n})\mathbb{Z}$. Then, the algebra of functions

$$\{f \in \mathrm{C}(X) \mid f(x) \in A_x \text{ for all } x \in X\},$$

is a norm-complete $u\ell$-group and every such a group can be represented in this way.

As a corollary we obtain

Corollary

The norm-completion of the algebra of $\mathbb{Z}$-maps is given by all continuous maps which respect denominators.

A duality for norm-complete MV-algebras

The category $\mathbb{MV}$

Let $\mathbb{MV}$ be the category whose objects are semisimple MV-algebras and arrows are MV-homomorphisms.

The category $\mathbb{A}$

Let $\mathbb{A}$ be the category whose objects are pairs $\langle X, \delta \rangle$, where X is a compact Hausdorff space and δ is a map from X into $\mathbb{N}$. An arrow between two objects $\langle X, \delta \rangle$ and $\langle Y, \delta' \rangle$ is a continuous map $f\colon X \to Y$ such that

$$\delta'(f(x)) \leq \delta(x).$$

A duality for norm-complete MV-algebras

The functor $\mathscr{L}$

Let $\mathscr{L}\colon \mathbb{A} \to \mathbb{MV}$ be an assignment that associates to every object $\langle X, \delta \rangle$ in $\mathbb{A}$ the MV-algebra

$$\mathscr{L}\left(\langle X, \delta \rangle\right) := \{g \in \mathtt{C}(X) \mid \forall x \in X \quad \mathtt{den}(g(x)) \mid \delta(x)\},$$

and to any $\mathbb{A}$-arrow $f\colon \langle X, \delta \rangle \to \langle Y, \delta' \rangle$ the $\mathbb{MV}$-arrow that sends each $h \in \mathscr{L}(\langle Y, \delta' \rangle)$ into the map $h \circ f$.

A duality for norm-complete MV-algebras

The functor $\mathscr{M}$

Let $\mathscr{M} : \mathbb{MV} \to \mathbb{A}$ be the assignment that associates to each MV-algebra A, the pair $\langle \mathtt{Max}(A), \delta_A \rangle$, where $\mathtt{Max}(A)$ is maximal spectrum of A and, for any $\mathfrak{m} \in \mathtt{Max}(A)$,

$$\delta_A(\mathfrak{m}) := \begin{cases} n & \text{if } A/\mathfrak{m} \text{ has } n+1 \text{ elements} \\ 0 & \text{otherwise.} \end{cases}$$

Let also $\mathscr{M}$ assign to every MV-homomorphism $h\colon A \to B$ the map that sends every $\mathfrak{m} \in \mathscr{M}(B)$ into its inverse image under h, in symbols $\mathscr{M}(h)(\mathfrak{m}) = h^{-1}[\mathfrak{m}] \in \mathtt{Max}(A)$.

A duality for norm-complete MV-algebras

Theorem

The functors $\mathscr{L}$ and $\mathscr{M}$ form a contravariant adjunction.

So, what is left to do in order to find a duality is to characterise the fixed points on each side.

It is quite easy to see the the fixed points on the algebraic side are exactly the norm-complete MV-algebras.

A duality for norm-complete MV-algebras

Conjecture

Let $\langle X, \delta \rangle$ be an object in $\mathbb{N}$. There exist a set J and a continuous embedding $\iota \colon X \to [0,1]^J$ such that $\mathtt{den}(\iota(x)) = \delta(x)$ if, and only if,

> *For pair of points $x, y \in X$ such that $x \neq y$, defining $d = \frac{1}{\delta(y)}$ if $\delta(y) \neq 0$ or $d = 1$ otherwise, there exists a family of open sets*
>
> $$\{O_q \mid q \in (0,d) \cap \mathbb{Q}\}$$
>
> *such that for any $p, q \in (0,d) \cap \mathbb{Q}$ and $n \in \mathbb{N}$*
>
> 1. *$p < q$ implies $\{x\} \subseteq O_p \subseteq \overline{O_p} \subseteq O_q \subseteq \overline{O_q} \subseteq \{y\}^c$.*
> 2. *$\delta^{-1}[\{n\}] \subseteq \bigcup \{O_p \mid \mathtt{den}(p) \mid n\}$.*

Sketch of the proof

The key step in the proof is to show that there are enough good functions to separate points:

Theorem

Let $\langle X, \delta \rangle$ be an $\mathbb{N}$-space satisfying the aforementioned condition. For any pair of distinct points $x, y \in X$ there exists a denominator respecting, continuous function $f\colon X \to [0,1]$ such that

$$f(x) = 0 \text{ and } f(y) = \begin{cases} \frac{1}{\delta(y)} & \text{if } \delta(y) \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}.$$

Sketch of the proof

Then we can use

Theorem (Kelley's Embedding lemma)

Let X and Y be topological spaces and $\mathcal{F}$ be a family of functions from X to Y. Suppose that all functions in $\mathcal{F}$ are continuous and that they separate points. Then the evaluation map $\mathrm{ev}\colon X \to Y^{\mathcal{F}}$ given by

$$\mathrm{ev}(x) = (f(x))_{f \in \mathcal{F}}$$

is continuous and injective.

It is immediate to see that if all functions in $\mathcal{F}$ respect denominators, then so does ev. Finally, since for all $x \in X$, the value $\frac{1}{\delta(x)}$ is attained by some f on x, in fact the function ev preserves δ.

Sketch of the proof

Theorem

Let X be a compact Hausdorff space and $\tilde{\delta}\colon X \hookrightarrow [0,1]^{\kappa}$ be a homeomorphism of X into its image. Let $\delta(x) = \mathtt{den}(\tilde{\delta}(x))$. The pair $\langle X, \delta\rangle$ is in $\mathbb{A}$ and is a-separated.

This completes the proof that $\langle X, \delta\rangle$ can be embedded in some $[0,1]^J$ preserving the prescriptions given by δ if, and only if, it is a-separated.

Kakutani duality, for groups

Corollary (of the conjecture)

The category of norm-complete archimedean $u\ell$-groups is dually equivalent to the full subcategory of $\mathbb{A}$ given by all a-separated spaces.

Thanks for your attention!