DENOMINATOR RESPECTING MAPS

Based on a joint work V. Marra (University of Milano).

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Embedding spaces

It is well known that

*every compact Hausdorff space $X$ can be embedded in some hypercube $[0, 1]^J$ for some index set $J$.*

Suppose that $X$ is now endowed with a function $\delta : X \to \mathbb{N}$.

**Problem**

Given a pair $\langle X, \delta \rangle$, *is there a continuous embedding $\iota : X \to [0, 1]^J$ in such a way that the denominators of the points in $\iota[X]$ agree with $\delta$?*

Let us assume that “agree” means that $\delta(x) = \text{den}(\iota(x))$. 
Recall that $\mathbb{N}$ forms a complete lattice under the divisibility order: the top being 0 and the bottom being 1.

Let $J$ be a set and $\bar{p} \in [0, 1]^J$. If $\bar{p} \in \mathbb{Q}^J$ we define its denominator to be the natural number

$$\text{den}(\bar{p}) = \text{lcm}\{p_i \mid i \in J\}$$

where lcm stands for the least common denominator. If $\bar{p} \notin \mathbb{Q}^J$ we set $\text{den}(\bar{p}) = 0$.

1. A function $f: [0, 1]^J \to [0, 1]$ preserves denominators if for any $\bar{x} \in [0, 1]^J$, $\text{den}(f(\bar{x})) = \text{den}(\bar{x})$.

2. A function $f: [0, 1]^J \to [0, 1]$ respects denominators if for any $\bar{x} \in [0, 1]^J$, $\text{den}(f(\bar{x})) | \text{den}(\bar{x})$. 
An easy counter-example

Consider $X = [0, 1]$ with its Euclidean topology and endow it with a constant $\delta$:

$$\forall x \in X \quad \delta(x) = 1.$$ 

The only points with denominator equal 1 in $[0, 1]^J$ are the so-called lattice points i.e., points whose coordinates are either 0 or 1.

The only way $\nu$ could agree with $\delta$ is to send all points in one lattice point —failing injectivity— or by sending the points in different lattice points —failing continuity.
The above mentioned problem is crucial in the duality theory of **MV-algebras** — the equivalent algebraic semantics of Łukasiewicz logic. An MV-algebra is a structure \( \langle A, \oplus, \neg, 0 \rangle \) such that

1. \( \langle A, \oplus, 0 \rangle \) is a commutative monoid,
2. \( \neg \neg x = x \),
3. \( \neg 0 \oplus x = \neg 0 \)
4. \( \neg (\neg x \oplus y) \oplus y = \neg (\neg y \oplus x) \oplus x \).

**Example**

The interval \([0, 1]\) in the real numbers has a natural MV-structure given by the **truncated sum** \( x \oplus y = \min\{x + y, 1\} \) and \( \neg x = 1 - x \). The importance of this structure comes from the fact that it generates the whole variety of MV-algebras.
**Theorem (Marra, S. 2012)**

Semisimple MV-algebras with their homomorphisms form a category that is dually equivalent to the category of compact Hausdorff spaces embedded in some hypercube, with \( \mathbb{Z} \)-maps among them.

**Definition**

For \( I, J \) arbitrary sets, a map from \( \mathbb{R}^I \) into \( \mathbb{R}^J \) is called \( \mathbb{Z} \)-map if it is continuous and piecewise (affine) linear map, where each (affine) linear piece has integer coefficients.

**Remark**

Since every \( \mathbb{Z} \)-map \( f \) acts on each point as an linear function with integer coefficients, it respect denominators i.e.,

\[
\text{den}(f(x)) \mid \text{den}(x).
\]
**Mundici’s functor**

An abelian \( l \)-group with order unit (\( ul \)-group, for short), is a partially ordered Abelian group \( G \) whose order is a lattice, and that possesses an element \( u \) such that

for all \( g \in G \), there exists \( n \in \mathbb{N} \) such that \( (n)u \geq g \).

The functor \( \Gamma \) that takes an \( ul \)-group \( \langle G, u \rangle \) to its unital interval \([0, u]\) with operation \( \oplus \) and \( \neg \) defined as follows:

\[
x \oplus y = \min\{u, x + y\} \quad \text{and} \quad \neg x = u - x,
\]

is full, faithful, and dense hence it has a quasi-inverse \( \Xi \) and

**Theorem (Mundici 1986)**

The pair \( \Gamma, \Xi \) gives an equivalence of categories between the category of MV-algebras with their morphisms, and the category of \( ul \)-groups with ordered group morphisms preserving the order unit.
Norm induced by the order unit

**Definition**

Let \((G, u)\) be a \(ul\)-group. The order unit \(u\) induces a **seminorm** \(\| \cdot \|_u\) defined as follows:

\[
\|g\|_u := \inf \left\{ \frac{p}{q} \in \mathbb{Q} \mid p, q \in \mathbb{N}, q \neq 0 \text{ and } q|g| \leq pu \right\}
\]

The seminorm \(\| \cdot \|_u : G \to \mathbb{R}^+\) is in fact a norm if, and only if, \(G\) is archimedean. **Any semisimple MV-algebra** \(A\) inherits a norm from its enveloping (archimedean) group \(\Xi(A)\).

**Definition**

An **norm-complete MV-algebra** is a semisimple MV-algebra which is Cauchy-complete w.r.t. its induced norm.
**Stone-Yosida-Kakutani duality**

**Theorem (Stone-Yosida-Kakutani duality 1941)**

A unital real vector space \((V, u)\) is isomorphic to \((C(X), 1)\) for some compact Hausdorff space \(X\), if, and only if, \(V\) is archimedean and norm-complete (with respect to the norm \(\|u\|\) induced by the unit).

**Question**

What if we want to substitute \(ul\)-group for real vector space in the above statements?

**Remark**

An answer was already given by Stone: compact Hausdorff spaces correspond to archimedean, complete and divisible \(ul\)-groups.
Denominator preserving maps

**Theorem (Goodearl-Handelman 1980)**

Let $X$ be a compact Hausdorff space. For any $x \in X$ choose $A_x$ to be either $A_x = \mathbb{R}$ or $A_x = (\frac{1}{n})\mathbb{Z}$. Then, the algebra of functions

$$\{ f \in C(X) \mid f(x) \in A_x \text{ for all } x \in X \},$$

is a norm-complete $u\ell$-group and every such a group can be represented in this way.

As a corollary we obtain

**Corollary**

The norm-completion of the algebra of $\mathbb{Z}$-maps is given by all continuous maps which respect denominators.
A duality for norm-complete MV-algebras

The category $\mathbf{MV}$

Let $\mathbf{MV}$ be the category whose objects are semisimple MV-algebras and arrows are MV-homomorphisms.

The category $\mathbf{A}$

Let $\mathbf{A}$ be the category whose objects are pairs $\langle X, \delta \rangle$, where $X$ is a compact Hausdorff space and $\delta$ is a map from $X$ into $\mathbb{N}$. An arrow between two objects $\langle X, \delta \rangle$ and $\langle Y, \delta' \rangle$ is a continuous map $f: X \to Y$ such that

$$\delta'(f(x)) \leq \delta(x).$$
The functor \( \mathcal{L} \)

Let \( \mathcal{L} : A \to \text{MV} \) be an assignment that associates to every object \( \langle X, \delta \rangle \) in \( A \) the MV-algebra

\[
\mathcal{L} (\langle X, \delta \rangle) := \{ g \in C(X) \mid \forall x \in X \quad \text{den}(g(x)) \mid \delta(x) \},
\]

and to any \( A \)-arrow \( f: \langle X, \delta \rangle \to \langle Y, \delta' \rangle \) the \( \text{MV} \)-arrow that sends each \( h \in \mathcal{L}(\langle Y, \delta' \rangle) \) into the map \( h \circ f \).
A duality for norm-complete MV-algebras

The functor \( \mathcal{M} \)

Let \( \mathcal{M} : \text{MV} \to \text{A} \) be the assignment that associates to each MV-algebra \( A \), the pair \( \langle \text{Max}(A), \delta_A \rangle \), where \( \text{Max}(A) \) is maximal spectrum of \( A \) and, for any \( m \in \text{Max}(A) \),

\[
\delta_A(m) := \begin{cases} 
  n & \text{if } A/m \text{ has } n + 1 \text{ elements} \\
  0 & \text{otherwise}. 
\end{cases}
\]

Let also \( \mathcal{M} \) assign to every MV-homomorphism \( h : A \to B \) the map that sends every \( m \in \mathcal{M}(B) \) into its inverse image under \( h \), in symbols \( \mathcal{M}(h)(m) = h^{-1}[m] \in \text{Max}(A) \).
A duality for norm-complete MV-algebras

Theorem

The functors $\mathcal{L}$ and $\mathcal{M}$ form a contravariant adjunction.

So, what is left to do in order to find a duality is to characterise the fixed points on each side.

It is quite easy to see the fixed points on the algebraic side are exactly the norm-complete MV-algebras.
A duality for norm-complete MV-algebras

Conjecture

Let \( \langle X, \delta \rangle \) be an object in \( \mathbb{N} \). There exist a set \( J \) and a continuous embedding \( \nu : X \to [0, 1]^J \) such that \( \text{den} (\nu(x)) = \delta(x) \) if, and only if,

For pair of points \( x, y \in X \) such that \( x \neq y \), defining \( d = \frac{1}{\delta(y)} \) if \( \delta(y) \neq 0 \) or \( d = 1 \) otherwise, there exists a family of open sets

\[
\{ O_q \mid q \in (0, d) \cap \mathbb{Q} \}
\]

such that for any \( p, q \in (0, d) \cap \mathbb{Q} \) and \( n \in \mathbb{N} \)

1. \( p < q \) implies
\[
\{ x \} \subseteq O_p \subseteq \overline{O_p} \subseteq O_q \subseteq \overline{O_q} \subseteq \{ y \}^c.
\]
2. \( \delta^{-1}[\{ n \}] \subseteq \bigcup \{ O_p \mid \text{den}(p) \mid n \} \).
Sketch of the proof

The key step in the proof is to show that there are enough good functions to separate points:

**Theorem**

Let \( \langle X, \delta \rangle \) be an \( \mathbb{N} \)-space satisfying the aforementioned condition. For any pair of distinct points \( x, y \in X \) there exists a denominator respecting, continuous function \( f: X \to [0, 1] \) such that

\[
f(x) = 0 \quad \text{and} \quad f(y) = \begin{cases} 
\frac{1}{\delta(y)} & \text{if } \delta(y) \in \mathbb{N} \\
1 & \text{otherwise}
\end{cases}.
\]
Sketch of the proof

Then we can use

**Theorem (Kelley’s Embedding lemma)**

Let $X$ and $Y$ be topological spaces and $\mathcal{F}$ be a family of functions from $X$ to $Y$. Suppose that all functions in $\mathcal{F}$ are *continuous* and that they separate points. Then the evaluation map $\text{ev} : X \rightarrow Y^\mathcal{F}$ given by

$$\text{ev}(x) = (f(x))_{f \in \mathcal{F}}$$

is continuous and injective.

It is immediate to see that if all functions in $\mathcal{F}$ respect denominators, then so does $\text{ev}$. Finally, since for all $x \in X$, the value $\frac{1}{\delta(x)}$ is attained by some $f$ on $x$, in fact the function $\text{ev}$ preserves $\delta$. 
Sketch of the proof

**Theorem**

Let $X$ be a compact Hausdorff space and $\tilde{\delta}: X \hookrightarrow [0, 1]^\kappa$ be a homeomorphism of $X$ into its image. Let $\delta(x) = \text{den}(\tilde{\delta}(x))$. The pair $\langle X, \delta \rangle$ is in $\mathbb{A}$ and is $a$-separated.

This completes the proof that $\langle X, \delta \rangle$ can be embedded in some $[0, 1]^J$ preserving the prescriptions given by $\delta$ if, and only if, it is $a$-separated.
Kakutani duality, for groups

Corollary (of the conjecture)

The category of norm-complete archimedean $ul$-groups is **dually equivalent** to the full subcategory of $A$ given by all $a$-separated spaces.

Thanks for your attention!