Baker-Beynon duality beyond semisimplicity

Università degli Studi di Salerno

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Part I

A generalization of Baker-Beynon duality

Baker-Beynon duality

Definition

- An (abelian) ℓ-group is an abelian group A equipped with a lattice order such that $a \leq b$ implies $a + c \leq b + c$ for every $a, b, c \in A$.
- A Riesz space V is an $\mathbb{R}$-vector space equipped with a lattice order such that it is an ℓ-group and $0 \leq r$ and $0 \leq v$ imply $rv \geq 0$ for each $r \in \mathbb{R}$ and $v \in V$.

ℓ-groups and Riesz spaces can be axiomatized by equations, and so they form varieties.

Definition

- A map between ℓ-groups is an ℓ-group homomorphism if it is a group and a lattice homomorphism.
- An ℓ-group homomorphism between Riesz spaces is a Riesz space homomorphism if it is a linear map.

Examples of Riesz spaces

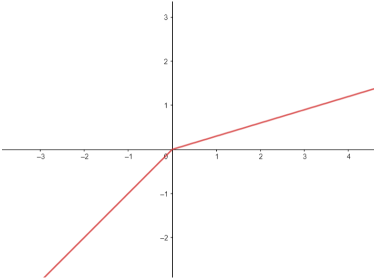
- $\mathbb{R}$
- $\mathbb{R}^X$ for a set X
- $\mathbb{R} \overrightarrow{\times} \mathbb{R}$ (lexicographic product)
- $C(X, \mathbb{R})$ for a topological space X
- $L^p(\mathbb{R}^n)$

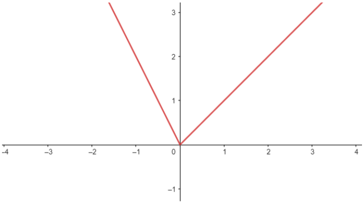
Examples of ℓ-groups

- All the examples above
- $\mathbb{Z}$
- $\mathbb{Z}^X$ for a set X
- $\mathbb{Z} \overrightarrow{\times} \mathbb{Z}$
- $\mathbb{Q}$

Definition

- A continuous function $f\colon \mathbb{R}^\kappa \to \mathbb{R}$ is piecewise linear (homogeneous) if there exist $g_1, \ldots, g_n\colon \mathbb{R}^\kappa \to \mathbb{R}$ linear homogeneous functions (each in finitely many variables) such that for each $x \in \mathbb{R}^\kappa$ we have $f(x) = g_i(x)$ for some $i = 1, \ldots, n$.
- We say that a piecewise linear function f has integer coefficients, if it is defined by $g_1, \ldots, g_n$ with integer coefficients.





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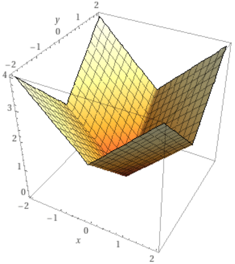
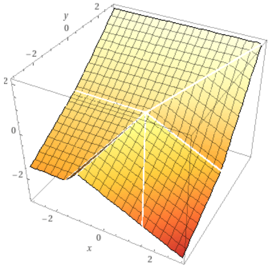




We denote by

- $\mathrm{PWL}(\mathbb{R}^\kappa)$ the Riesz space of piecewise linear functions $f\colon \mathbb{R}^\kappa \to \mathbb{R}$;
- $\mathrm{PWL}_\mathbb{Z}(\mathbb{R}^\kappa)$ the ℓ-group of piecewise linear functions $f\colon \mathbb{R}^\kappa \to \mathbb{R}$ with integer coefficients.

Theorem (Baker 1968)

Let κ be a cardinal number.

- *The free Riesz space on κ generators is isomorphic to* $\mathrm{PWL}(\mathbb{R}^\kappa)$.
- *The free ℓ-group on κ generators is isomorphic to* $\mathrm{PWL}_\mathbb{Z}(\mathbb{R}^\kappa)$.

- The element $[t]$ of the free algebra correspond to the piecewise linear function that maps $x \in \mathbb{R}^\kappa$ to $t(x) \in \mathbb{R}$.
- The free generators of the free algebra correspond to the projections maps onto each coordinate.

If $X \subseteq \mathbb{R}^\kappa$, we denote

- $\mathrm{PWL}(X) = \{f|_X \text{ with } f \in \mathrm{PWL}(\mathbb{R}^\kappa)\}$,
- $\mathrm{PWL}_\mathbb{Z}(X) = \{f|_X \text{ with } f \in \mathrm{PWL}_\mathbb{Z}(\mathbb{R}^\kappa)\}$.

Which Riesz spaces (ℓ-groups) are isomorphic to $\mathrm{PWL}(X)$ ($\mathrm{PWL}_\mathbb{Z}(X)$) for some $X \subseteq \mathbb{R}^\kappa$?

Congruences in ℓ-groups and Riesz spaces correspond to ℓ-ideals.

Definition

An ℓ-ideal in a Riesz space (ℓ-group) is a subgroup I that is convex, i.e. $|a| \leq |b|$ and $b \in I$ imply $a \in I$.

ℓ-ideals in Riesz spaces are automatically vector subspaces.

Definition

- A proper ℓ-ideal is called maximal if it is maximal wrt inclusion.
- A nontrivial Riesz space (ℓ-group) A is simple if $\{0\}$ and A are the only ℓ-ideals of A.
- A Riesz space (ℓ-group) is semisimple if the intersection of all its maximal ℓ-ideals is $\{0\}$.

Proposition

- *An ℓ-group is simple iff it embeds into $\mathbb{R}$.*
- *A Riesz space is simple iff it is isomorphic to $\mathbb{R}$.*
- *An ℓ-group or Riesz space is semisimple iff it can be embedded into a power of $\mathbb{R}$.*

Examples

- $\mathbb{R} \overrightarrow{\times} \mathbb{R}$ and $\mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ with the lexicographic order are not semisimple (and hence not simple).
- $\mathbb{R}$ is simple as a ℓ-group and as a Riesz space.
- $\mathbb{Z}$ and $\mathbb{Q}$ are simple ℓ-groups.
- $C(X, \mathbb{R})$ is semisimple for any topological space X.
- $\mathrm{PWL}(X)$ and $\mathrm{PWL}_{\mathbb{Z}}(X)$ are semisimple for any $X \subseteq \mathbb{R}^{\kappa}$.

Theorem (Baker 1968)

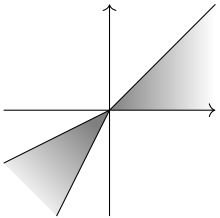
- *Every semisimple Riesz space is isomorphic to* $\mathrm{PWL}(X)$ *for some* $X \subseteq \mathbb{R}^\kappa$.
- *Every semisimple* ℓ*-group is isomorphic to* $\mathrm{PWL}_{\mathbb{Z}}(X)$ *for some* $X \subseteq \mathbb{R}^\kappa$.

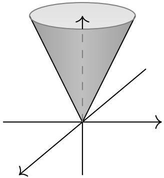
Theorem (Baker 1968)

- *Every semisimple Riesz space is isomorphic to* $\mathrm{PWL}(C)$ *for some closed cone* $C \subseteq \mathbb{R}^\kappa$.
- *Every semisimple* ℓ*-group is isomorphic to* $\mathrm{PWL}_{\mathbb{Z}}(C)$ *for some closed cone* $C \subseteq \mathbb{R}^\kappa$.

Definition

A nonempty subset $C \subseteq \mathbb{R}^\kappa$ is a closed cone if it is closed under multiplication by nonnegative scalars and it is closed in $\mathbb{R}^\kappa$ with the euclidean topology.





Let $\mathscr{F}_\kappa$ be the free Riesz space (ℓ-group) over κ generators.
For any $T \subseteq \mathscr{F}_\kappa$ and $S \subseteq \mathbb{R}^\kappa$, we define the following operators.

$$\mathrm{V}(T) = \{x \in \mathbb{R}^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\}$$
$$\mathrm{I}(S) = \{[t] \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\}.$$

Galois connection

$$T \subseteq \mathrm{I}(S) \quad \text{iff} \quad S \subseteq \mathrm{V}(T)\,.$$

- $\mathrm{V}(T)$ is always a closed cone of $\mathbb{R}^\kappa$.
- $\mathrm{I}(S)$ is always an ℓ-ideal of $\mathscr{F}_\kappa$.

What are the fixpoints of the Galois connection?

$S = \mathsf{V}\,\mathsf{I}(S)$ iff S is a closed cone in $\mathbb{R}^\kappa$.

S is a fixpoint iff $S = \mathsf{V}(T)$ for some $T \subseteq \mathscr{F}_\kappa \cong \mathrm{PWL}(\mathbb{R}^\kappa)$. It can be shown that closed cones are exactly the vanishing sets of families of piecewise linear functions (with integer coefficients) on $\mathbb{R}^\kappa$.

$T = \mathsf{I}\,\mathsf{V}(T)$ iff T is a ℓ-ideal of $\mathscr{F}_\kappa$ that is intersection of maximal ℓ-ideals.

T is a fixpoint iff $T = \mathsf{I}(S)$ for some $S \subseteq \mathbb{R}^\kappa$ iff $T = \bigcap\{\mathsf{I}(x) \mid x \in S\}$. The proper ℓ-ideals of the form $\mathsf{I}(x)$ for some $x \in \mathbb{R}^\kappa$ are exactly the maximal ideals of $\mathscr{F}_\kappa$ (follows from the characterization of simple algebras).

Proposition

The poset of ℓ-ideals of $\mathscr{F}_\kappa$ that are intersections of maximal ℓ-ideals is dually isomorphic to the poset of closed cones in $\mathbb{R}^\kappa$.

We can extend this dual isomorphism to a dual equivalence of categories between the category of semisimple Riesz spaces (ℓ-groups) and the category of closed cones.

On objects:

Let A be a semisimple Riesz space (ℓ-group), then $A \cong \mathscr{F}_\kappa / J$, where J is an intersection of maximal ℓ-ideals of $\mathscr{F}_\kappa$. Then map

$$A \mapsto \mathsf{V}(J),$$

where $\mathsf{V}(J)$ is a closed cone in $\mathbb{R}^\kappa$.

Let C be a closed cone in $\mathbb{R}^\kappa$. Then map

$$C \mapsto \mathsf{PWL}(C),$$

which is semisimple and isomorphic to $\mathscr{F}_\kappa / \mathsf{I}(C)$. (In the case of ℓ-groups map C to $\mathsf{PWL}_{\mathbb{Z}}(C)$.)

On morphisms:

Let $h\colon A \to B$ be a Riesz space (ℓ-group) homomorphism with $A \cong \mathscr{F}_\kappa / J_A$ and $B \cong \mathscr{F}_\mu / J_B$. Then map

$$h \mapsto f_h,$$

with $f_h\colon \mathrm{V}(J_B) \to \mathrm{V}(J_A)$ the piecewise linear map whose i^{th} component is given by $h([a_i]) \in \mathscr{F}_\mu / J_B$ where a_i is the i^{th} generator of $\mathscr{F}_\kappa$.

Let $f\colon C \to D$ be a piecewise linear function (with integer coefficients) between closed cones. Then map

$$f \to h_f,$$

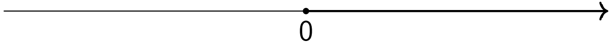
where $h_f\colon \mathrm{PWL}(D) \to \mathrm{PWL}(C)$ is given by $h_f(g) = g \circ f$ (in the case of ℓ-groups we have $h_f\colon \mathrm{PWL}_{\mathbb{Z}}(D) \to \mathrm{PWL}_{\mathbb{Z}}(C)$).

These functors yield the Baker-Beynon duality:

Theorem (Beynon 1974)

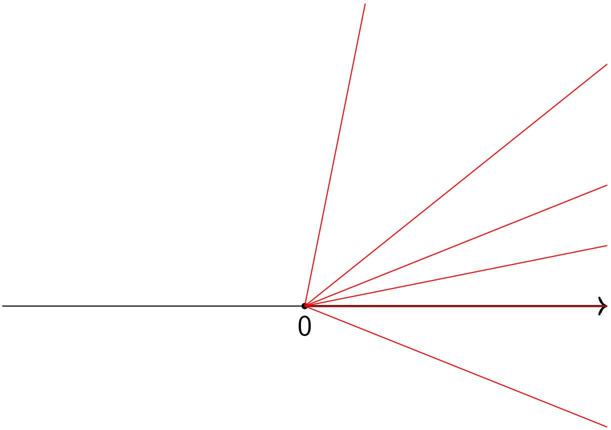
- *The category of semisimple Riesz spaces is dually equivalent to the category of closed cones in $\mathbb{R}^\kappa$ and piecewise linear maps with real coefficients.*
- *The category of semisimple ℓ-groups is dually equivalent to the category of closed cones in $\mathbb{R}^\kappa$ and piecewise linear maps with integer coefficients.*

$\mathbb{R}$ (as a Riesz space) is dual to the semiline $\{x \in \mathbb{R} \mid x \geq 0\}$.

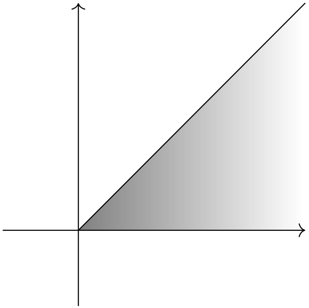


$\mathbb{R}$ (as a Riesz space) is dual to the semiline $\{x \in \mathbb{R} \mid x \geq 0\}$.

Indeed, $\mathbb{R} \cong \mathrm{PWL}(\{x \in \mathbb{R} \mid x \geq 0\})$.



$\mathscr{F}_2 / \langle (x - y) \wedge y \wedge 0 \rangle$ is dual to $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x\}$.



Generalizing Baker-Beynon duality beyond semisimplicity

In the definition of the operators

$$\begin{aligned} \mathsf{V}(T) =& \{x \in \mathbb{R}^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\} \text{ with } T \subseteq \mathscr{F}_\kappa \\ \mathsf{I}(S) =& \{[t] \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\} \text{ with } S \subseteq \mathbb{R}^\kappa. \end{aligned}$$

we can replace $\mathbb{R}$ with any Riesz space (ℓ-group) A and still get a Galois connection.

In the definition of the operators

$$\begin{aligned} \mathsf{V}(T) =& \{x \in A^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\} \text{ with } T \subseteq \mathscr{F}_\kappa \\ \mathsf{I}(S) =& \{[t] \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\} \text{ with } S \subseteq A^\kappa. \end{aligned}$$

we can replace $\mathbb{R}$ with any Riesz space (ℓ-group) A and still get a Galois connection.

Caramello, Marra, and Spada (2021) observed that this can be done for any variety of algebras by replacing $\mathbb{R}$ with any algebra in that variety. They also show that this approach also works in a more categorical setting.

Our goal is to replace $\mathbb{R}$ with a Riesz space that guarantees more ℓ-ideals of $\mathscr{F}_\kappa$ to be fixpoints of $\mathsf{I}\mathsf{V}$. In this way we extend Baker-Beynon duality beyond semisimple Riesz spaces and ℓ-groups.

It is not possible to obtain a Riesz space (ℓ-group) A such that for any κ the fixpoints of $\mathbf{I}\mathbf{V}$ are all the ℓ-ideals of $\mathscr{F}_\kappa$. This is a consequence of the fact that there are subdirectly irreducible Riesz spaces (ℓ-groups) of arbitrarily large cardinality.

However, if we fix a cardinal α, we will see that we can find A such that for any $\kappa < \alpha$ the fixpoints of $\mathbf{I}\mathbf{V}$ are all the ideals of $\mathscr{F}_\kappa$.

We will see how this yields a duality for all Riesz spaces (ℓ-groups) that are κ-generated (i.e. generated by a set of cardinality at most κ) with $\kappa < \alpha$. In particular, we obtain a duality for all finitely generated Riesz spaces (ℓ-groups) by taking $\alpha = \omega$.

We will replace maximal ℓ-ideals with prime ℓ-ideals.

Definition

An ℓ-ideal I is prime if $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Theorem

- *A/I is linearly ordered iff I is prime.*
- *Every ℓ-ideal is intersection of prime ℓ-ideals.*
- *Every Riesz space (ℓ-group) is subdirect product of linearly ordered ones.*

We fix a cardinal α and we look for a Riesz space (ℓ-group) A into which all the κ-generated with $\kappa < \alpha$ linearly ordered Riesz spaces (ℓ-groups) embed.

Theorem (C., Lapenta, Spada)

Let α be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ in which all κ-generated (with $\kappa < \alpha$) linearly ordered Riesz spaces and ℓ-groups embed.

Proof sketch.

- The theory of nontrivial linearly ordered Riesz spaces is complete. So, each lin. ordered Riesz space $A \neq 0$ is elementarily equivalent to $\mathbb{R}$.
- Thus, for any cardinal β there is an ultrapower of $\mathbb{R}$ into which all the linearly ordered Riesz spaces of cardinality less than β embed.
- Since a Riesz space that is κ-generated has cardinality at most $\max(\kappa, 2^\omega)$, it is sufficient to take $\beta = \max(\alpha, 2^\omega)$. □

For $\alpha = \omega$ we have can pick $\mathcal{U}$ as follows:

Proposition

Let $\mathcal{U}$ be any ultrapower of $\mathbb{R}$ over a nonprincipal ultrafilter of a countably infinite set. Then every finitely generated linearly ordered Riesz space and ℓ-group embeds into $\mathcal{U}$.

Fix a cardinal α and $\mathcal{U}$ an ultrapower of $\mathbb{R}$ in which all κ-generated with $\kappa < \alpha$ linearly ordered Riesz spaces and ℓ-groups embed. κ will denote an arbitrary cardinal smaller than α.
We consider the operators:

$$\mathrm{V}(T) = \{x \in \mathcal{U}^\kappa \mid t(x) = 0 \text{ for all } [t] \in T\} \text{ with } T \subseteq \mathscr{F}_\kappa$$
$$\mathrm{I}(S) = \{[t] \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\} \text{ with } S \subseteq \mathcal{U}^\kappa.$$

Galois connection

$$T \subseteq \mathrm{I}(S) \quad \text{iff} \quad S \subseteq \mathrm{V}(T)\,.$$

- $T = \mathrm{I}\,\mathrm{V}(T)$ iff T is an ℓ-ideal of $\mathscr{F}_\kappa$.
- We call $S \subseteq \mathcal{U}^\kappa$ such that $S = \mathrm{V}\,\mathrm{I}(S)$ a generalized closed cone ($\mathbb{Z}$-generalized closed cone).

Proposition

The poset of ℓ-ideals of $\mathscr{F}_\kappa$ is dually isomorphic to the poset of generalized closed cones ($\mathbb{Z}$-generalized closed cones) in $\mathcal{U}^\kappa$.

Definition

- We say that a map $\mathcal{U}^\kappa \to \mathcal{U}^\mu$ is definable ($\mathbb{Z}$-definable) if its components are defined by terms in the language of Riesz spaces (ℓ-groups).
- If $X \subseteq \mathcal{U}^\kappa$, we denote by $\mathrm{Def}(X)$ and $\mathrm{Def}_\mathbb{Z}(X)$ the sets of definable maps and $\mathbb{Z}$-definable maps $f\colon X \to \mathcal{U}$.

The functors $A \cong \mathscr{F}_\kappa / J \mapsto \mathrm{V}(J)$ and $C \mapsto \mathrm{Def}(C) \cong \mathscr{F}_\kappa / \mathrm{I}(C)$ induce:

Theorem (C., Lapenta, Spada)

- *The category of κ-generated Riesz spaces (with $\kappa < \alpha$) is dually equivalent to the category of generalized closed cones in $\mathcal{U}^\kappa$ (with $\kappa < \alpha$) and definable maps.*

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- We say that a map $\mathcal{U}^\kappa \to \mathcal{U}^\mu$ is definable ($\mathbb{Z}$-definable) if its components are defined by terms in the language of Riesz spaces (ℓ-groups).
- If $X \subseteq \mathcal{U}^\kappa$, we denote by $\mathrm{Def}(X)$ and $\mathrm{Def}_{\mathbb{Z}}(X)$ the sets of definable maps and $\mathbb{Z}$-definable maps $f : X \to \mathcal{U}$.

The functors $A \cong \mathscr{F}_\kappa / J \mapsto \mathrm{V}(J)$ and $C \mapsto \mathrm{Def}_{\mathbb{Z}}(C) \cong \mathscr{F}_\kappa / \mathrm{I}(C)$ induce:

Theorem (C., Lapenta, Spada)

- *The category of κ-generated Riesz spaces (with $\kappa < \alpha$) is dually equivalent to the category of generalized closed cones in $\mathcal{U}^\kappa$ (with $\kappa < \alpha$) and definable maps.*
- *The category of κ-generated ℓ-groups (with $\kappa < \alpha$) is dually equivalent to the category of $\mathbb{Z}$-generalized closed cones in $\mathcal{U}^\kappa$ (with $\kappa < \alpha$) and $\mathbb{Z}$-definable maps.*

Consequences and applications of the duality

Proposition

- *The generalized closed cones in $\mathcal{U}^\kappa$ (together with $\varnothing$) form the closed of a topology on $\mathcal{U}^\kappa$. The closure of a nonempty $X \subseteq \mathcal{U}^\kappa$ is $\mathsf{V}\,\mathsf{I}(X)$.*
- *$\mathbb{R}^\kappa$ is a subset of $\mathcal{U}^\kappa$ and the closed subsets of $\mathbb{R}^\kappa$ with the subspace topology are exactly the closed cones (and $\varnothing$).*

We obtain the following correspondences:

$\mathscr{F}_\kappa$	$\mathbb{R}^\kappa$	$\mathcal{U}^\kappa$
maximal ℓ-ideals	half-lines from the origin	closures of points of $\mathbb{R}^\kappa$ (except the origin)
intersections of maximal ℓ-ideals	closed cones	closures of nonempty subsets of $\mathbb{R}^\kappa$
prime ℓ-ideals		irreducible closed subsets $=$ closures of points of $\mathcal{U}^\kappa$ (except the origin)
ℓ-ideals		generalized closed cones

If A is a Riesz space (ℓ-group), then $\mathtt{Spec}(A) = \{\text{prime } \ell\text{-ideals of } A\}$ is called the spectrum of A and is naturally equipped with the Zariski topology generated by the closed subsets $\{P \in \mathtt{Spec}(A) \mid a \in P\}$, where a ranges in A.

If P is a prime ℓ-ideal of $\mathscr{F}_\kappa$, then $\mathrm{V}(P)$ is the closure of a point of $\mathcal{U}^\kappa$. Choose one such point $x_P \in \mathcal{U}^\kappa$ for each $P \in \mathtt{Spec}(\mathscr{F}_\kappa)$. Let $\mathscr{E} \colon \mathtt{Spec}(\mathscr{F}_\kappa) \to \mathcal{U}^\kappa$ be defined by $\mathscr{E}(P) = x_P$.

Theorem (C., Lapenta, Spada)

- $\mathscr{E}$ *is a topological embedding.*
- $\mathscr{E}^{-1}$ *is a complete lattice isomorphism between* $\mathrm{Op}(\mathcal{U}^\kappa \setminus \{O\})$ *and* $\mathrm{Op}(\mathtt{Spec}(\mathscr{F}_\kappa))$.

The spectrum of each Riesz space (ℓ-group) is a generalized spectral space, i.e. it is T_0, sober, the compact open subsets form a basis, and the intersection of two compact opens is compact.

Theorem (C., Lapenta, Spada)

$\mathcal{U}^\kappa \setminus \{O\}$ *is a generalized spectral space.*

$\mathscr{E}\colon \mathtt{Spec}(\mathscr{F}_\kappa) \to \mathcal{U}^\kappa$ can be thought of as a coordinatization of $\mathtt{Spec}(\mathscr{F}_\kappa)$ with coordinates in $\mathcal{U}$.

By the correspondence theorem, if $A \cong \mathscr{F}_\kappa/J$, then we can think of $\mathtt{Spec}(A)$ as a subspace of $\mathtt{Spec}(\mathscr{F}_\kappa)$. So, $\mathscr{E}$ restricts to an embedding of $\mathtt{Spec}(A)$ into $\mathcal{U}^\kappa$ whose image is $\mathscr{E}[\mathtt{Spec}(\mathscr{F}_\kappa)] \cap \mathsf{V}(J)$.

While the spectrum as a topological space is not sufficient to recover the original Riesz space, the coordinatization is enough:

Theorem (C., Lapenta, Spada)

$A \cong \mathrm{Def}(\mathscr{E}[\mathtt{Spec}(A)])$ *for any Riesz space A.*

An analogous result holds for ℓ-groups.

In part II we will see how $\mathscr{E}[\mathtt{Spec}(\mathscr{F}_\kappa)]$ looks like when κ is finite.

Recall that a Riesz space (ℓ-group) is semisimple if the intersection of all its maximal ℓ-ideals is $\{0\}$.

Theorem (C., Lapenta, Spada)

Let A be a Riesz space (ℓ-group) and $C \subseteq \mathcal{U}^\kappa$ its dual generalized closed cone ($\mathbb{Z}$-generalized closed cone).
A is semisimple iff $C = \mathbb{V}\,\mathbb{I}(C \cap \mathbb{R}^\kappa)$, i.e. C is the closure of $C \cap \mathbb{R}^\kappa$ in $\mathcal{U}^\kappa$.

Note that $C \cap \mathbb{R}^\kappa$ is the closed cone in $\mathbb{R}^\kappa$ corresponding to A under Baker-Beynon duality.

Let A, B be two Riesz spaces (ℓ-groups) dual to the generalized closed cones ($\mathbb{Z}$-generalized closed cones) $C \subseteq U^{\kappa}$ and $D \subseteq \mathcal{U}^{\mu}$.

Theorem

- *The product* $A \times B$ *is dual to* $(C \times \{O\}) \cup (\{O\} \times D) \subseteq \mathcal{U}^{\kappa+\mu}$.
- *The coproduct* $A \oplus B$ *is dual to* $C \times D \subseteq \mathcal{U}^{\kappa+\mu}$.
- *The lexicographic product* $\mathbb{R} \overrightarrow{\times} B$ *(*$\mathbb{Z} \overrightarrow{\times} B$ *in the case of* ℓ*-groups) is dual to*

$$\{(x, y) \in \mathcal{U} \times D \mid 0 < x,\ y/x \text{ has all infinitesimal coordinates}\} \cup \{O\}.$$

Part II

Using non-standard tools

From now on we will assume $\alpha = \omega$.

Let also assume that $\mathcal{U}$ is an ultrapower of $\mathbb{R}$ defined as $\mathcal{U} = \mathbb{R}^{\mathbb{N}}/\mathcal{F}$ with $\mathcal{F}$ a nonprincipal ultrafilter of $\mathcal{P}(\mathbb{N})$.

We have seen that $\mathcal{U}$ induces dualities for finitely generated Riesz spaces and ℓ-groups.

Theorem

- *The category of all finitely generated Riesz spaces is dually equivalent to the category of generalized closed cones in $\mathcal{U}^n$ (with $n \in \mathbb{N}$).*
- *The category of all finitely generated ℓ-groups is dually equivalent to the category of $\mathbb{Z}$-generalized closed cones in $\mathcal{U}^n$ (with $n \in \mathbb{N}$).*

It follows from Łoś's theorem that the algebraic structure of $\mathbb{R}$ lifts to $\mathcal{U}$:

Proposition

- *$\mathcal{U}$ is a linearly ordered field.*
- *$\mathcal{U}^n$ is a $\mathcal{U}$-vector space.*

The elements of $\mathcal{U}$ are equivalence classes $[(r_i)_{i\in\mathbb{N}}]$ of $\mathbb{N}$-indexed sequences $(r_i)_{i\in\mathbb{N}}$ of real numbers. Where

$$(r_i)_{i\in\mathbb{N}} \sim (s_i)_{i\in\mathbb{N}} \quad \text{iff} \quad \{i \in \mathbb{N} \mid r_i = s_i\} \in \mathscr{F} .$$

We identify each $r \in \mathbb{R}$ with $[(r_i)_{i\in\mathbb{N}}] \in \mathcal{U}$ such that $r_i = r$ for all $i \in \mathbb{N}$.

Proposition

- *$\mathbb{R}$ embeds into $\mathcal{U}$ as a sub-lattice-ordered field.*
- *$\mathcal{U}^n$ is an $\mathbb{R}$-vector space containing $\mathbb{R}^n$ as a vector subspace.*

We will identify $\mathbb{R}$ and $\mathbb{R}^n$ with their isomorphic copies in $\mathcal{U}$ and $\mathcal{U}^n$.

Some notions from non-standard analysis

As it is common in non-standard analysis, we call the elements of $\mathcal{U}$ hyperreal numbers. Among the hyperreal numbers we have:

- real numbers

$$[(1,1,1,\dots)], \quad \left[\left(\frac{15}{7},\frac{15}{7},\frac{15}{7},\dots\right)\right], \quad [(\pi,\pi,\pi,\dots)],\dots$$

- infinitesimal numbers (absolute value smaller than any $0 < r \in \mathbb{R}$)

$$\left[\left(1,\frac{1}{2},\frac{1}{3},\dots\right)\right], \quad \left[\left(1,\frac{1}{2^2},\frac{1}{3^2},\dots\right)\right], \quad \left[\left(1,\frac{1}{2^2},\frac{1}{2^3},\dots\right)\right],\dots$$

- unlimited numbers (absolute value greater than any $r \in \mathbb{R}$)

$$[(1,2,3,\dots)], \quad \left[\left(1,2^2,3^2,\dots\right)\right], \quad \left[\left(1,2^2,2^3,\dots\right)\right],\dots$$

- limited numbers (not limited, i.e. between $-r$ and r for some $r \in \mathbb{R}$)

$$[(1,1,1,\dots)], \quad \left[\left(1,\frac{1}{2},\frac{1}{3},\dots\right)\right], \quad \left[\left(\frac{1}{2},\frac{2}{3},\frac{3}{4},\dots,1-\frac{1}{n},\dots\right)\right],\dots$$

The operations behave like the limits in analysis:

- limited $+$ limited $=$ limited, unlimited $+$ limited $=$ unlimited, ...
- limited $\times$ limited $=$ limited, unlimited $\times$ infinitesimal $=$?, ...

Definition

- If $A \subseteq \mathbb{R}^n$, its enlargement ${}^*A \subseteq \mathcal{U}^n$ is defined as follows:

$$\Big([(r_i^1)], \dots, [(r_i^n)]\Big) \in {}^*A \text{ if and only if } \{i \in \mathbb{N} \mid (r_i^1, \dots, r_i^n) \in A\} \in \mathcal{F}.$$

- If $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$, then the enlargement ${}^*f : {}^*A \to \mathcal{U}$ of f is given by

$$^*f([(r_i^1)], \dots, [(r_i^n)]) := [(f(r_i^1, \dots, r_i^n))].$$

Proposition

- *$A \subseteq {}^*A$.*
- *If A is finite, then $A = {}^*A$.*
- *If A is infinite, then ${}^*A$ must contain some elements of $\mathcal{U}^n$ outside $\mathbb{R}^n$.*

For example, ${}^*\mathbb{N}$ contains the unlimited element $[(1, 2, 3, \dots)]$.

Let $\mathscr{L}$ be a first-order language and $(\mathbb{R}, (P_\alpha), (f_\alpha)))$ an $\mathscr{L}$-structure, where the P_α's and f_α's are the interpretations of the predicate and function symbols of $\mathscr{L}$ in $\mathbb{R}$. Then $(\mathcal{U}, ({}^*P_\alpha), ({}^*f_\alpha)))$ is also an $\mathscr{L}$-structure.

Theorem (Transfer principle)

Let φ be a first-order $\mathscr{L}$-sentence. Then φ is true in $(\mathbb{R}, (P_\alpha), (f_\alpha))$ if and only φ is true in $(\mathcal{U}, ({}^*P_\alpha), ({}^*f_\alpha))$.

In other words, a first-order condition holds in $\mathbb{R}$ iff the condition obtained by replacing all the relations and functions with their enlargements holds in $\mathcal{U}$. (For simplicity of notation, we just write $+$ instead of ${}^*+$ and similarly for the other lattice-ordered field operations.)

This allows to transfer first-order properties of functions and subsets from $\mathbb{R}^n$ to $\mathcal{U}^n$ and back.

Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$.
Since

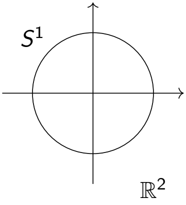
$$\forall x, y\,((x, y) \in S^1 \Leftrightarrow x^2 + y^2 = 1)$$

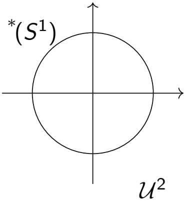
is a first-order condition that holds in $\mathbb{R}$, then

$$\forall x, y\,((x, y) \in {}^*(S^1) \Leftrightarrow x^2 + y^2 = 1)$$

holds in $\mathcal{U}$ by transfer. So, ${}^*(S^1) = \{(x, y) \in \mathcal{U}^2 \mid x^2 + y^2 = 1\}$.

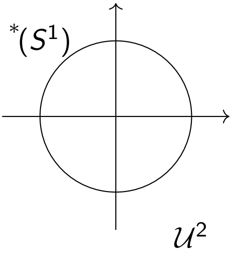
It is easy to get a geometric intuition of the enlargements of subsets of $\mathbb{R}^n$ defined by first-order sentences.



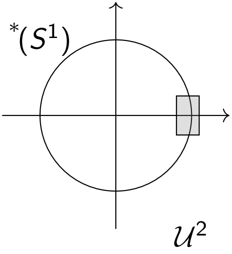


If $0 < \varepsilon \in \mathcal{U}$ is infinitesimal, then $x = \left(\frac{1}{\sqrt{1+\varepsilon^2}}, \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \right) \in {}^*(S^1) \setminus S^1$.

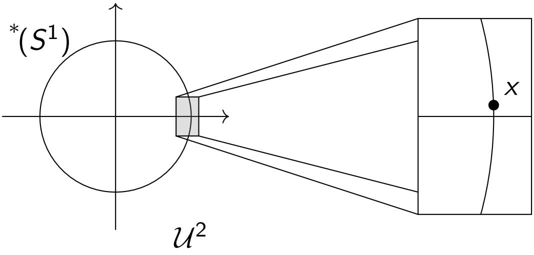
If $0 < \varepsilon \in \mathcal{U}$ is infinitesimal, then $x = \left(\frac{1}{\sqrt{1+\varepsilon^2}}, \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \right) \in {}^*(S^1) \setminus S^1$.



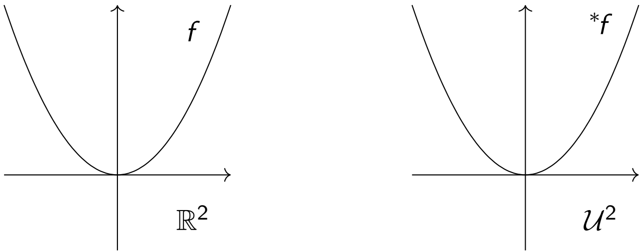
If $0 < \varepsilon \in \mathcal{U}$ is infinitesimal, then $x = \left(\frac{1}{\sqrt{1+\varepsilon^2}}, \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \right) \in {}^*(S^1) \setminus S^1$.



If $0 < \varepsilon \in \mathcal{U}$ is infinitesimal, then $x = \left(\frac{1}{\sqrt{1+\varepsilon^2}}, \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}\right) \in {}^*(S^1) \setminus S^1$.



If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a function, then the graph of ${}^*f\colon \mathcal{U}^n \to \mathcal{U}$ is just the enlargement of the graph of f.



The enlargement of f can be used to compute limits. For example,

$$\lim_{x\to 0} f(x) = 0 \iff {}^*f(x) \text{ infinitesimal for all } x \text{ infinitesimal.}$$

Definable maps and piecewise linear functions

Let $g\colon \mathcal{U}^n \to \mathcal{U}$ be definable, i.e. there is a term t such that $g(x) = t(x)$ for all $x \in \mathcal{U}^n$.

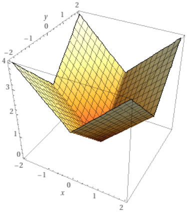
If $f\colon \mathbb{R}^n \to \mathbb{R}$ is the piecewise linear function defined by the same term, i.e. $f(x) = t(x)$ for all $x \in \mathbb{R}^n$, then the transfer principle yields

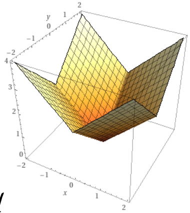
$$\forall x \in \mathbb{R}^n (f(x) = t(x)) \quad \text{iff} \quad \forall x \in \mathcal{U}^n ({}^*f(x) = t(x)).$$

Thus, $g = {}^*f$, and so g is the enlargement of a piecewise linear function.

Proposition

Let $C \subseteq \mathcal{U}^n$ be a generalized closed cone. Then $\mathrm{Def}(C) = \{({}^*f)_{|C} \mid f\colon \mathbb{R}^n \to \mathbb{R}$ *piecewise linear*$\}$.

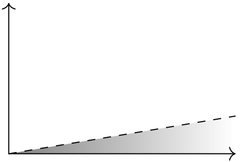
$f\colon \mathbb{R}^2 \to \mathbb{R}$ 

${}^*f\colon \mathcal{U}^2 \to \mathcal{U}$ 

Definable functions naturally generalize piecewise linear functions.

Let $\mathbb{R}\overrightarrow{\times}\mathbb{R}$. Then its dual generalized closed cone is

$$C = \{(x, y) \in \mathcal{U}^2 \mid x > 0,\ y \geq 0,\ \text{and } y/x \text{ is infinitesimal}\} \cup \{(0, 0)\}.$$



So,

$$\begin{aligned} \mathbb{R}\overrightarrow{\times}\mathbb{R} \cong \mathrm{Def}(C) &= \{({}^*f)_{|C} \mid f \colon \mathbb{R}^2 \to \mathbb{R} \text{ piecewise linear}\} \\ &= \{({}^*f)_{|C} \mid f \colon \mathbb{R}^2 \to \mathbb{R} \text{ linear}\}. \end{aligned}$$

Indexes and irreducible closed subsets

Recall from part I that f.g. linearly ordered Riesz spaces correspond to the irreducible closed subsets of $\mathcal{U}^n$, i.e. the closures of the points of $\mathcal{U}^n$.

We want to understand how these subsets of $\mathcal{U}^n$ look like (for simplicity we only consider the case of Riesz spaces).

Theorem (Orthogonal decomposition)

If $x \in \mathcal{U}^n$, then $x = \alpha_1 v_1 + \cdots + \alpha_k v_k$ where $\alpha_1, \ldots, \alpha_k \in \mathcal{U}$ are positive, α_{i+1}/α_i is infinitesimal for each $i < k$, and $v_1, \ldots, v_k \in \mathbb{R}^n$ are orthonormal vectors. Furthermore, this decomposition is unique.

Definition

- We call a finite sequence $(v_1, \ldots, v_k)$ of orthonormal vectors in $\mathbb{R}^n$ an index.
- We denote by $\iota(x)$ the index $(v_1, \ldots, v_k)$ made of the vectors appearing in the orthogonal decomposition of $x \in \mathcal{U}^n$.
- Let $\mathbf{v}, \mathbf{w}$ be two indexes. We write $\mathbf{v} \leq \mathbf{w}$ when $\mathbf{v}$ is a truncation of $\mathbf{w}$, i.e. $\mathbf{v} = (v_1, \ldots, v_h)$ and $\mathbf{w} = (v_1, \ldots, v_k)$ for $h \leq k$.

Definition

If $\mathbf{v}$ is an index, let $\mathrm{Cone}(\mathbf{v}) := \{y \in \mathcal{U}^n \mid \iota(y) \leq \mathbf{v}\}$

Theorem (C., Lapenta, Spada)

The closure of x in $\mathcal{U}^n$ is $\mathrm{Cone}(\iota(x))$.

The proof uses the fact that if $f : \mathbb{R}^n \to \mathbb{R}$ is a linear function and $x \in \mathcal{U}^n$ with $\iota(x) = (v_1, \ldots, v_k)$, then the sign of ${}^*f(x)$ is determined by the real numbers $f(v_1), \ldots, f(v_k)$.

Corollary

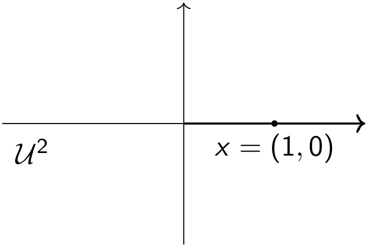
If $x \in \mathcal{U}^n$, then

$$\begin{aligned}\mathrm{Def}(\mathrm{Cone}(\iota(x))) &\cong \{{}^*f(x) \in \mathcal{U} \mid f : \mathbb{R}^n \to \mathbb{R} \textit{ piecewise linear}\} \\ &= \{{}^*f(x) \in \mathcal{U} \mid f : \mathbb{R}^n \to \mathbb{R} \textit{ linear}\}.\end{aligned}$$

Let $x = (1,0) \in \mathcal{U}^2$. Then $\iota(x) = (v_1)$ with $v_1 = (1,0)$. We have

$$y \in \text{Cone}(\iota(x)) \quad \text{iff} \quad y = \alpha_1(1,0) \text{ with } 0 \leq \alpha_1 \in \mathcal{U}.$$

Thus, the closure of x in $\mathcal{U}^2$ is $\{(\alpha_1, 0) \mid 0 \leq \alpha_1 \in \mathcal{U}\}$, which is the enlargement of the positive x-semiaxis.



The dual Riesz space is $\mathbb{R}$. Indeed,

$$\text{Def}(\text{Cone}(\iota(x))) \cong \{{}^*f(1,0) \mid f : \mathbb{R}^2 \to \mathbb{R} \text{ linear}\} \cong \mathbb{R}.$$

Let $\varepsilon \in \mathcal{U}$ be a positive infinitesimal and $x = (1, \varepsilon)$. Then

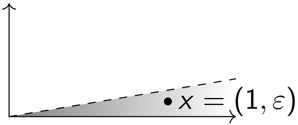
$$x = 1(1,0) + \varepsilon(0,1)$$

is the orthogonal decomposition of x. Thus, $\iota(x) = (v_1, v_2)$ with $v_1 = (1,0)$ and $v_2 = (0,1)$. We have

$$\begin{aligned} y \in \mathrm{Cone}(\iota(x)) \quad \text{iff} \quad & y = O, \text{ or} \\ & y = \alpha_1(1,0) \text{ (orthogonal decomposition), or} \\ & y = \alpha_1(1,0) + \alpha_2(0,1) \text{ (orthogonal decomposition)} \end{aligned}$$

Then $\mathrm{Cone}(\iota(x))$, i.e. the closure of x in $\mathcal{U}^2$ is

$$\{(\alpha_1, \alpha_2) \in \mathcal{U}^2 \mid \alpha_1 > 0,\ \alpha_2 \geq 0 \text{ and } \alpha_2/\alpha_1 \text{ is infinitesimal}\} \cup \{O\}.$$



The dual Riesz space is $\mathbb{R} \overrightarrow{\times} \mathbb{R}$. Indeed,

$$\begin{aligned} \mathrm{Def}(\mathrm{Cone}(\iota(x))) &\cong \{{}^*f(1,\varepsilon) \mid f \colon \mathbb{R}^n \to \mathbb{R} \text{ linear}\} \\ &= \{a + b\varepsilon \in \mathcal{U} \mid a, b \in \mathbb{R}\} \cong \mathbb{R} \overrightarrow{\times} \mathbb{R}. \end{aligned}$$

Theorem (C., Lapenta, Spada)

The mapping $\mathrm{Cone}\colon \mathbf{v} \mapsto \mathrm{Cone}(\mathbf{v})$ *induces an order-isomorphism between the set of indexes ordered by truncation and the set of irreducible closed subsets of* $\mathcal{U}^n$ *ordered by inclusion.*

Corollary

$\mathrm{I} \circ \mathrm{Cone}\colon \mathbf{v} \mapsto \mathrm{I}(\mathrm{Cone}(\mathbf{v}))$ *induces an order-isomorphism between the set of nonempty indexes ordered by truncation and* $\mathtt{Spec}(\mathscr{F}_n)$ *ordered by reverse inclusion.*

That nonempty indexes correspond to prime ideals of $\mathscr{F}_n$ was proved by Panti (1999) using different techniques.

Embedding $\mathtt{Spec}(\mathscr{F}_n)$ into $\mathcal{U}^n$

Recall from part I: if we choose for each irreducible closed subset $C \subseteq \mathcal{U}^n \setminus \{O\}$ a point $x \in \mathcal{U}^n$ such that C is the closure of x, then we can define an embedding $\mathscr{E} \colon \mathtt{Spec}(\mathscr{F}_n) \to \mathcal{U}^n$.

Indexes allow us to choose x for every C in a canonical way. Fix a positive infinitesimal $\varepsilon \in \mathcal{U}$. If $C = \text{Cone}(\mathbf{v})$ is an irreducible closed with $\mathbf{v} = (v_1, \ldots, v_k)$, then we pick $x \in \text{Cone}(\mathbf{v})$ defined as

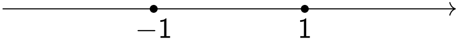
$$x = v_1 + \varepsilon v_2 + \cdots + \varepsilon^{k-1} v_k.$$

Since $\mathbf{v} = \iota(x)$, we have that $\text{Cone}(\mathbf{v})$ is the closure of x.

Therefore, we obtain an embedding $\mathscr{E} \colon \mathtt{Spec}(\mathscr{F}_n) \to \mathcal{U}^n$ that maps a prime ideal $P = \text{I}(\text{Cone}(\mathbf{v}))$ to the point $v_1 + \varepsilon v_2 + \cdots + \varepsilon^{k-1} v_k$.

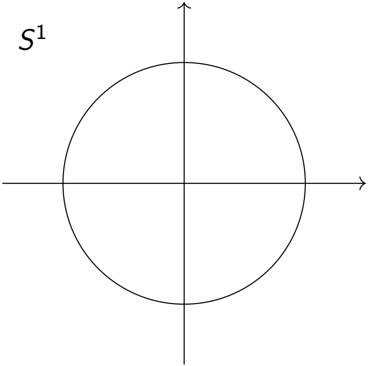
$\mathtt{Spec}(\mathscr{F}_1)$

We have $\mathscr{E}[\mathtt{Spec}(\mathscr{F}_1)] = \{-1, 1\} \subseteq \mathcal{U}$.

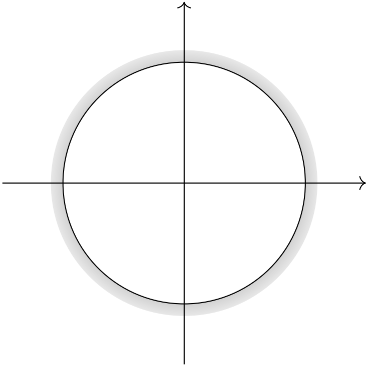


Note that $\mathtt{Spec}(\mathscr{F}_1) = \mathtt{MaxSpec}(\mathscr{F}_1)$.

We have $\mathscr{E}[\mathtt{MaxSpec}(\mathscr{F}_2)] = S^1 \subseteq \mathbb{R}^2 \subseteq \mathcal{U}^2$.

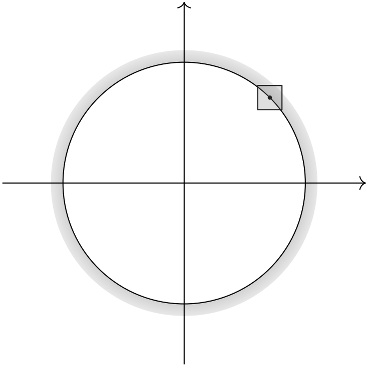


We have $\mathscr{E}[\mathtt{Spec}(\mathscr{F}_2)] \subseteq \mathcal{U}^2$ consists of points infinitesimally close to S^1.



$\mathtt{Spec}(\mathscr{F}_2)$

We have $\mathscr{E}[\mathtt{Spec}(\mathscr{F}_2)] \subseteq \mathcal{U}^2$ consists of points infinitesimally close to S^1.



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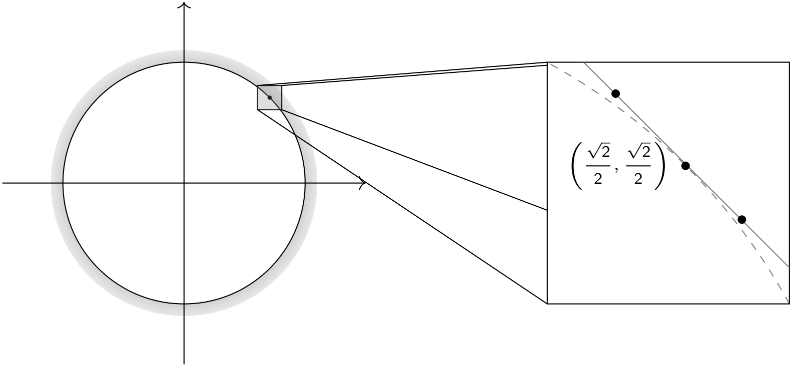

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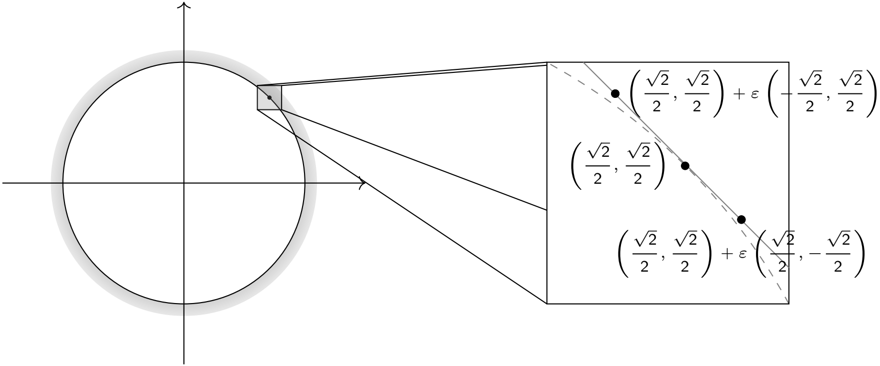
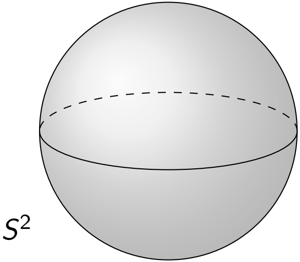


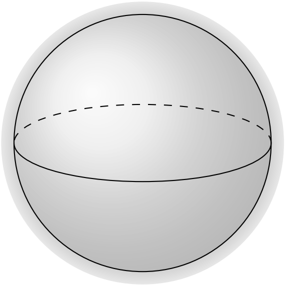
$\texttt{Spec}(\mathscr{F}_3)$

We have $\mathscr{E}[\texttt{MaxSpec}(\mathscr{F}_3)] = S^2 \subseteq \mathbb{R}^3 \subseteq \mathcal{U}^3$.



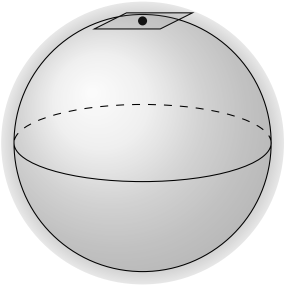
$\mathtt{Spec}(\mathscr{F}_3)$

We have $\mathscr{E}[\mathtt{Spec}(\mathscr{F}_3)] \subseteq \mathcal{U}^3$ consists of points infinitesimally close to S^2.



Spec($\mathscr{F}_3$)

We have $\mathscr{E}[\mathrm{Spec}(\mathscr{F}_3)] \subseteq \mathcal{U}^3$ consists of points infinitesimally close to S^2.



Spec($\mathscr{F}_3$)

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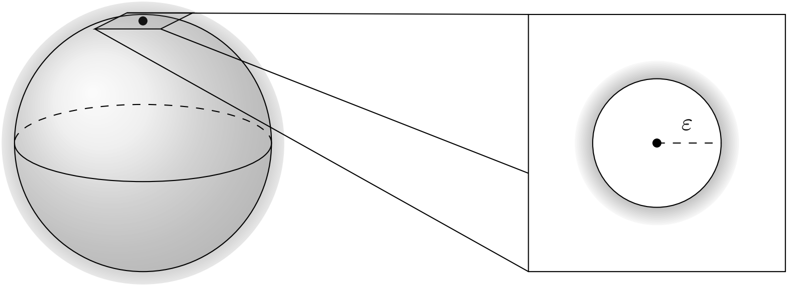


Bonus slides

Characterization of prime ideals in $\mathscr{F}_n$

We have seen that $\mathsf{I} \circ \mathsf{Cone}$ induces an order-isomorphism between indexes and prime ideals of $\mathscr{F}_n$. Recall that $\mathscr{F}_n \cong \mathrm{PWL}(\mathbb{R}^n)$

$\mathsf{I}(\mathsf{Cone}(\mathbf{v}))$ correspond to the prime ideal of $\mathrm{PWL}(\mathbb{R}^n)$ given by

$$\{f \in \mathrm{PWL}(\mathbb{R}^n) \mid {}^*f \text{ vanishes on } \mathsf{Cone}(\mathbf{v})\}.$$

Is there a way to avoid mentioning the enlargement?

Definition

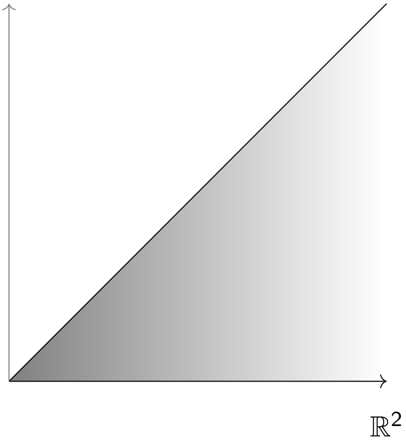
Let $\mathbf{v} = (v_1, \dots, v_k)$ be an index. We say that a closed cone $C \subseteq \mathbb{R}^n$ is a $\mathbf{v}$-*cone* if there exist real numbers $0 < r_1, \dots, r_k \in \mathbb{R}$ such that C is the positive span of

$$\{r_1 v_1, \quad r_1 v_1 + r_2 v_2, \quad \dots \quad r_1 v_1 + \dots + r_k v_k\}.$$

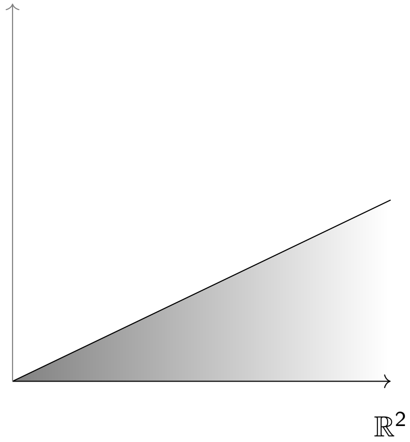
Theorem (C., Lapenta, Spada)

$\mathsf{Cone}(\mathbf{v})$ *is the intersection of the enlargements of all* $\mathbf{v}$-*cones.*

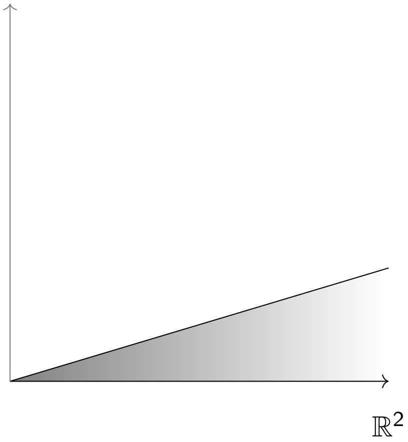
Let $\mathbf{v} = (v_1, v_2)$ with $v_1 = (1, 0)$ and $v_2 = (0, 1)$.



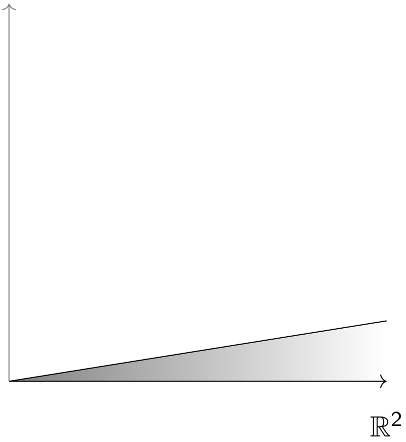
Let $\mathbf{v} = (v_1, v_2)$ with $v_1 = (1, 0)$ and $v_2 = (0, 1)$.



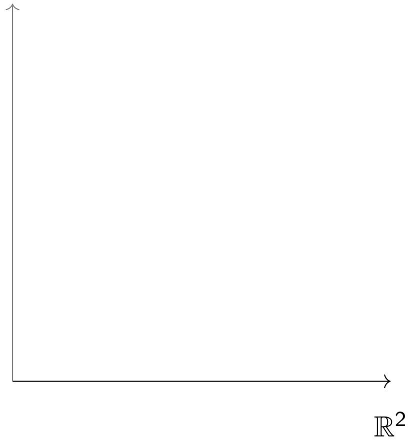
Let $\mathbf{v} = (v_1, v_2)$ with $v_1 = (1, 0)$ and $v_2 = (0, 1)$.



Let $\mathbf{v} = (v_1, v_2)$ with $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

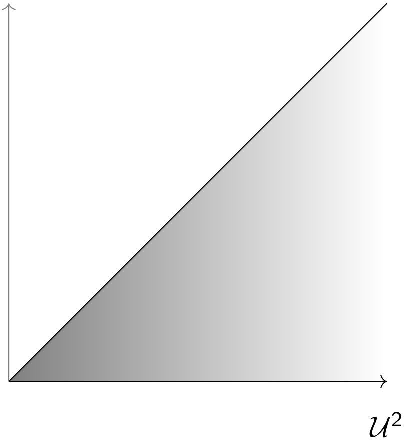


Let $\mathbf{v} = (v_1, v_2)$ with $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

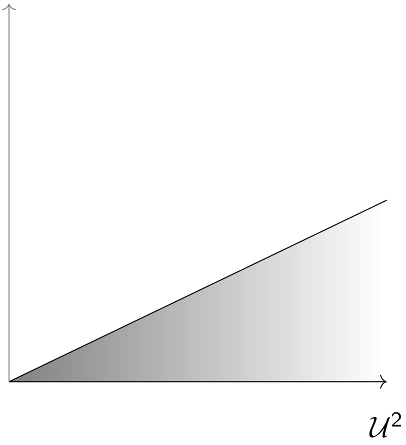


$\bigcap\{ C \mid C \text{ is a } \mathbf{v}\text{-cone}\}$ is the positive x-semiaxis.

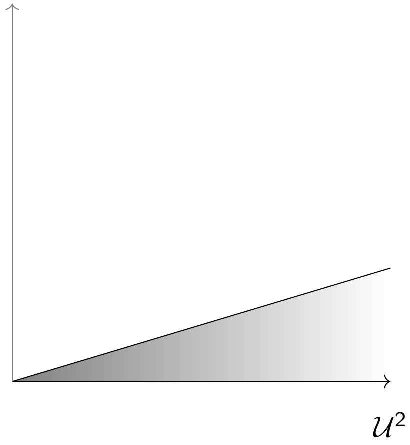
Let $\mathbf{v} = (v_1, v_2)$ with $v_1 = (1, 0)$ and $v_2 = (0, 1)$.



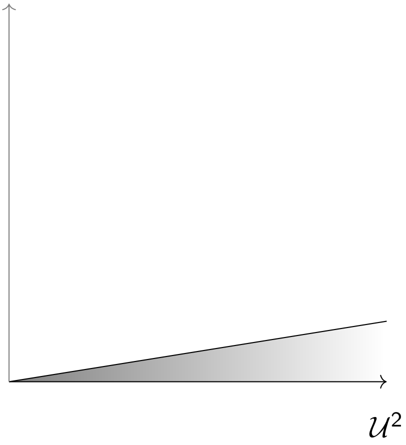
Let $\mathbf{v} = (v_1, v_2)$ with $v_1 = (1, 0)$ and $v_2 = (0, 1)$.



Let $\mathbf{v} = (v_1, v_2)$ with $v_1 = (1, 0)$ and $v_2 = (0, 1)$.



Let $\mathbf{v} = (v_1, v_2)$ with $v_1 = (1, 0)$ and $v_2 = (0, 1)$.



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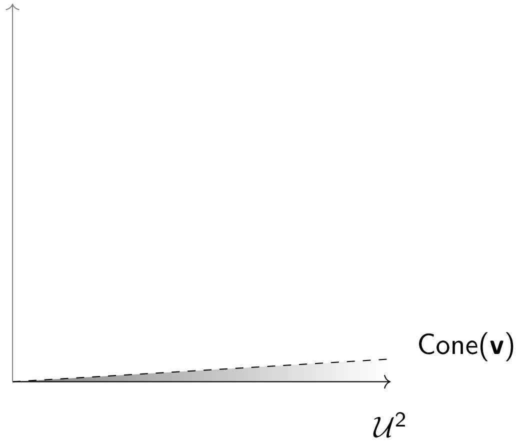


$$\bigcap\{{}^*C \mid C \text{ is a } \mathbf{v}\text{-cone}\} = \text{Cone}(\mathbf{v}).$$

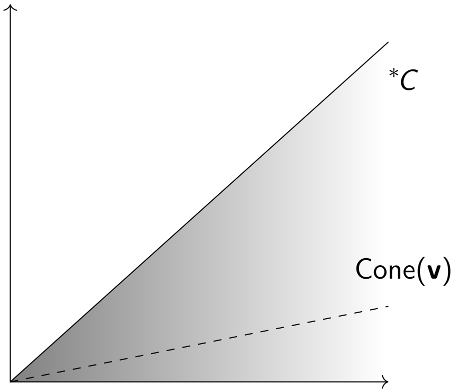
Theorem (C., Lapenta, Spada)

$^*f$ vanishes on $\mathrm{Cone}(\mathbf{v})$ *iff f vanishes on a* **v***-cone.*

Proof sketch.

- By transfer, if f vanishes on a **v**-cone C, then $^*f$ vanishes on $^*C$. So, $^*f$ vanishes on $\mathrm{Cone}(\mathbf{v})$ because $\mathrm{Cone}(\mathbf{v}) \subseteq {}^*C$.

□



$^*C$
Cone($\mathbf{v}$)

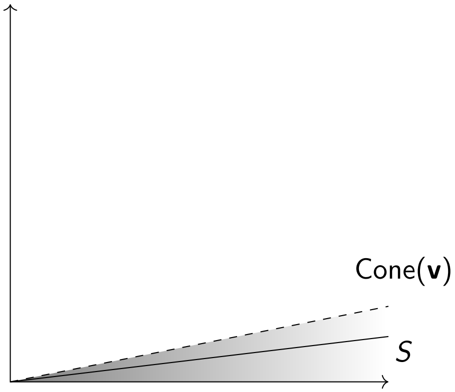
Theorem (C., Lapenta, Spada)

${}^*f$ vanishes on Cone($\mathbf{v}$) *iff f vanishes on a $\mathbf{v}$-cone.*

Proof sketch.

- By transfer, if f vanishes on a $\mathbf{v}$-cone C, then ${}^*f$ vanishes on ${}^*C$. So, ${}^*f$ vanishes on Cone($\mathbf{v}$) because Cone($\mathbf{v}$) $\subseteq {}^*C$.
- If ${}^*f$ vanishes on Cone($\mathbf{v}$), then there are $0 < \alpha_1, \ldots, \alpha_k \in \mathcal{U}$ such that the positive span S of $\{\alpha_1 v_1, \ldots, \alpha_1 v_1 + \cdots + \alpha_k v_k\}$, is contained in Cone($\mathbf{v}$).

□


Cone(**v**)
S

Theorem (C., Lapenta, Spada)

${}^*f$ vanishes on Cone($\mathbf{v}$) *iff f vanishes on a **v**-cone.*

Proof sketch.

- By transfer, if f vanishes on a **v**-cone C, then ${}^*f$ vanishes on ${}^*C$. So, ${}^*f$ vanishes on Cone($\mathbf{v}$) because Cone($\mathbf{v}$) $\subseteq {}^*C$.
- If ${}^*f$ vanishes on Cone($\mathbf{v}$), then there are $0 < \alpha_1, \ldots, \alpha_k \in \mathcal{U}$ such that the positive span S of $\{\alpha_1 v_1, \ldots, \alpha_1 v_1 + \cdots + \alpha_k v_k\}$, is contained in Cone($\mathbf{v}$).
- Thus, ${}^*f$ vanishes on S. By the transfer principle, there are $0 < r_1, \ldots, r_k \in \mathbb{R}$ such that f vanishes on the positive span of $\{r_1 v_1, \ldots, r_1 v_1 + \cdots + r_k v_k\}$, which is a **v**-cone.

□

We obtain the characterization of prime ideals of $\mathscr{F}_n$ due to Panti (1999).

Corollary

$\mathrm{I}(\mathrm{Cone}(\mathbf{v})) = \{f \in \mathrm{PWL}(\mathbb{R}^n) \mid f$ *vanishes on a **v**-cone*$\}$.

Additional slides

Theorem

Let $\alpha \leq 2^\omega$ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ such that every linearly ordered Riesz space an ℓ-groups of cardinality less than α embeds $\mathcal{U}$.

Proof.

- All nontrivial linearly ordered Riesz spaces are elementarily equivalent: their theory has quantifier elimination, and hence it is model complete. Since $\mathbb{R}$ embeds into every non-trivial Riesz space, the theory of linearly ordered Riesz Spaces is complete because it is model complete and has an algebraically prime model.
- By a model-theoretic fact any α-regular ultrapower $\mathcal{U}$ of $\mathbb{R}$ is such that all linearly ordered groups of cardinality less or equal to α embed into $\mathcal{U}$.
- Since $\alpha \leq 2^\omega$ (the cardinality of the language of Riesz spaces), another model theoretic fact tells us that every ℓ-group of cardinality less than α embeds into $\mathcal{U}$.

□

Definition (Panti (1999))

- We call a subspace of $\mathbb{R}^n$ rational if it admits a basis made of vectors from $\mathbb{Q}^n$.
- If $S \subseteq \mathbb{R}^n$, then its rational envelope $\langle S \rangle$ denotes the smallest rational subspace of $\mathbb{R}^n$ containing S
- We say that an index $\mathbf{v} = (v_1, \ldots, v_k)$ is $\mathbb{Z}$-reduced if $v_i \in \langle v_j \rangle^\perp$ for any $i \neq j$.

Given an index $\mathbf{v}$ there is a canonical way to associate a $\mathbb{Z}$-reduced index $\mathrm{red}(\mathbf{v})$.

Theorem (C., Lapenta, Spada)

- *The closure of x in $\mathcal{U}^n$ with the topology of the $\mathbb{Z}$-generalized closed cones is* $\bigcup\{\mathrm{Cone}(\mathbf{v}) \mid \mathrm{red}(\mathbf{v}) \leq \mathrm{red}(\iota(x))\}$.
- *There is an order isomorphism between $\mathbb{Z}$-reduced indexes and irreducible closed subsets of $\mathcal{U}^n$ with the topology of the $\mathbb{Z}$-generalized closed cones.*

Definition

Let A be a Riesz space (ℓ-group) and $0 < a \in A$.

- a is a strong order-unit if for each $b \in A$ there exists $n \in \mathbb{N}$ such that $b \leq na$.
- a is a weak order-unit of A if $a \wedge |b| = 0$ implies $b = 0$ for each $b \in A$.

Theorem

Let A be a nontrivial Riesz space (ℓ-group) and $C \subseteq \mathcal{U}^{\kappa}$ its dual generalized closed cone ($\mathbb{Z}$-generalized closed cone).

- *A has a strong order-unit iff $C \setminus \{O\}$ is compact.*
- *A has a weak order-unit iff $C \setminus \{O\}$ contains a dense compact open subset.*

For any natural number n let $\pi_n \colon \mathcal{U}^\omega \to \mathcal{U}^{n+1}$ be the map that sends $(x_i)_{i\in\omega}$ to $(x_0, x_1, \ldots, x_n)$.

Theorem (C., Lapenta, Spada)

Let A be an ω-generated Riesz space (ℓ-group) and $C \subseteq \mathcal{U}^\omega$ its dual generalized closed cone ($\mathbb{Z}$-generalized closed cone).
Then A is archimedean iff

$$C = \bigcap_{n=0}^{\infty} \pi_n^{-1}[\mathrm{V}\,\mathrm{I}(\mathrm{V}\,\mathrm{I}(\pi_n[C]) \cap \mathbb{R}^{n+1})],$$

where the subsets $\pi_n^{-1}[\mathrm{V}\,\mathrm{I}(\mathrm{V}\,\mathrm{I}(\pi_n[C]) \cap \mathbb{R}^{n+1})]$ form a decreasing sequence of generalized closed cones in $\mathcal{U}^\omega$.

When $\kappa > \omega$, the decreasing sequence is substituted by a downdirected family of generalized closed cones in $\mathcal{U}^\kappa$.