A uniform version of Di Nola Theorem

a joint work with A. Di Nola and G. Lenzi

Luca Spada
Dipartimento di Matematica e Informatica
Università di Salerno

http://logica.dmi.unisa.it/lucaspada

MV-algebras

MV-algebras are the equivalent algerbraic semantics of Lukasiewicz logic.

A structure $A = (A, \oplus, \neg, 0)$ is an MV-algebra if A satisfies the following equations, for every $x, y, z \in A$:

(i) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$; *(ii)* $x \oplus y = y \oplus x$;

(iii) $x \oplus 0 = x$; *(iv)* $x \oplus \neg 0 = \neg 0$;

(v) $\neg\neg x = x$; *(vi)* $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Representations of MV algebras

The main general tools in representation theory of MV-algebras are

- Chang representation Theorem,
- McNaughton Theorem and
- Di Nola representation Theorem.

The main result

The main result in this talk gives a non-standard representation of any MV-algebra A depending only on the cardinality of A.

Theorem: For any infinite cardinal α, there exists an ultrapower of $[0,1]^*$, such that all MV-algebras of cardinality smaller or equal than α embed in an MV-algebra of functions with values in $[0,1]^*$.

Controlling the cardinality

lemma. Let A be an MV-algebra and (G,u) an lu-group such that $A \approx \Gamma(G,u)$. Let α be an infinite cardinal then
$|A| = \alpha$ if, and only if, $|G| = \alpha$.

lemma. Let G be an abelian l-group and α be an infinite cardinal such that $|G| = \alpha$. Then **G can be embedded into an abelian divisible l-group D_G** such that **$|D_G| = \alpha$**.

α-regular filters

definition. Let α be a cardinal. A proper filter D over I is said to be ***α*-regular** if there exists a subset E of D such that $|E| = \alpha$ and each $i \in I$ belongs to only finitely many $e \in E$.

definition. Given a cardinal α, we say that a model A is ***α*-universal** if for every model B we have:

$$\mathfrak{B} \equiv \mathfrak{A} \quad \text{and} \quad |\mathfrak{B}| < \alpha \quad \text{implies} \quad \mathfrak{B} \hookrightarrow_{el.} \mathfrak{A}.$$

α^+-universal ultrapowers

Theorem. [Chang-Keisler] Let $|\mathcal{L}| \leq \alpha$ and D be an ultrafilter which is α-regular. Then, for every model A, the ultrapower $\Pi_D A$ is α^+-universal.

Lemma. For any sentence ψ the language of MV algebras there is a formula with only one free variable $\phi(v)$ in the language of lu-group such that for any MV-algebra A we have:

$$A \models \psi \quad \text{if, and only if,} \quad G \models \psi[u],$$

for any abelian l-group G and $u > 0$ in G such that $A \approx \Gamma(G,u)$.

The additive group of reals

Since any non-trivial divisible totally ordered l-group is elementarily equivalent to the real numbers seen as an additive group, from the previous result we get:

Proposition. Any non-trivial divisible MV-chain is elementarily equivalent to $\Gamma(R,1) = [0,1]$.

Representing MV-chains

Proposition. Let α be an infinite cardinal and A be an MV-chain such that $|A| = \alpha$. Then A can be embedded into an ultrapower of the MV-algebra [0,1] via an ultrafilter α-regular over α which does not depend on A.

A sketch of the proof

Proof. Let A be an infinite MV-chain such that $|A| = \alpha$ and $A \approx \Gamma(G,u)$. Then G is an ordered abelian group with strong unit u and $|G| = \alpha$.

So (G,u) can be embedded into a divisible ordered group D_G with strong unit u_D; in addition $|D_G| = \alpha$.

Now let $A_d \approx \Gamma(D_G, u_D)$: then A embeds in A_d and A_d is a divisible MV-algebra; so A_d is elementarily equivalent to [0,1].

Let F be a α-regular ultrafilter over α; then $\Pi_F[0,1]$ is α^+-universal, hence A_d embeds in $\Pi_F[0,1]$. Combining the embeddings we get that A can be embedded into the ultrapower $\Pi_F[0,1]$.

The general case

Proposition. Let A be an MV-algebra such that $|A| = \alpha$, with α an infinite cardinal. Then there exists a set X such that A can be embedded into an MV-algebra of functions from X to an ultrapower of the MV-algebra [0,1] via an α-regular ultrafilter over α which does not depend on A.

Proof.

$$A \hookrightarrow \Pi_{P \in Spec(A)} A/P.$$

Corollary. For any infinite cardinal α there exists a single MV-algebra of functions such that every MV-algebra of cardinality smaller or equal than α embeds into it.

Extimating the cardinality

It is also possible to give a sharp bound on the cardinality of the target algebra, based on following fact, which is part of the classical literature on the subject.

Proposition. [Chang-Keisler] Let F be a α-regular ultrafilter of α, with α infinite cardinal, then $|\Pi_F A| = |A|^{\alpha}$.

A “canonical” MV-algebra

The above construction gives no information on the target algebra, it only asserts its existence.

We will see now how it is possible to *construct* such an algebra in ZFC

The key tool in this construction are iterated ultrapowers.

Introducing Iterated Ultrapowers

An iterated ultrapower can be roughly described as a structure obtained from a linearly ordered set of ultrapowers and such that all these ultrapowers are embedded into it.

We sketch here such a construction:

- Let A be a first order structure
- I be a set
- $(X,<)$ a linear order
- $D=\langle D_x\rangle_{x\in X}$ a l.o. sequence of ultrafilters on I

Functions that live on a finite set

Let K=I^X be the set of all functions from X to I. Let Z be a subset of X. We say that a **function *f*** with domain K **lives on** Z if, for every function $i \in K$, $f(i)$ depends only on $i_{|Z}$.

We say that a **subset of *K* lives on** Z if its characteristic function lives on Z.

Let hereafter K be finite

The ultrafilter associated

To any Z we associate an ultrafilter D_Z on I^Z as follows:

$$D_Z = \{s \subseteq B_Z : D_{x1} y_1 \ldots D_{xn} y_n . \{(x_1, y_1), \ldots, (x_n, y_n)\} \in s\},$$

where $D_x y. \phi(y)$ means $\{y : \phi(y)\} \in D_x$.

Consider the set

$$E(D) = \{s \subseteq K : \exists Z.\ s \text{ lives on } Z \text{ and } s \downarrow Z \in D_Z\},$$

where $s \downarrow Z$ is the set of all restrictions to Z of the members of s.

Iterated Ultrapowers I

Proposition. [Chang-Keisler] The subsets of *K*, living on some finite subset of *X*, form a Boolean algebra *S*. The set E(D) is an ultrafilter in S.

E(D) can be considered as an infinitary product of the ultrafilters D_x (although it is not an ultrafilter on K as one could expect).

Iterated Ultrapowers II

Definition. Let A be a first order structure, I be a set, and D be a linearly ordered sequence of ultrafilters on I indexed by the linear order $(X,<)$. **The iterated ultrapower of A on D**, denoted $\Pi_D A$, is a first order structure over the same language as A.

The domain of Π_D A is the set of all functions f from $K(=B^X)$ to A which live on some finite subset of X, modulo the equivalence $=_D$ given by:

$$f =_D g \quad \text{if, and only if,} \quad \{i \in K : f(i)=g(i)\} \in E(D).$$

If R is any predicate symbol in the language of A then

$$R^A([f_1], \ldots [f_n]) \text{ iff } \{i \in K : R^A(f_1(i), \ldots f_n(i))\} \in E(D).$$

A final stratagem

Lemma. For every $x \in X$, Π_{Dx}A embeds elementarily in Π_DA.

So, for our aim, it is sufficient to find a definable linear order of all ultrafilter on a given ordinal.

V. Kanovei and S. Shelah. A definable nonstandard model of the reals. *Journal of Symbolic Logic*, **69**(1):159–164, 2004.

A final stratagem

Let P(α) be the powerset of α. P(α) has a natural "lexicographic" linear order: given E,F ⊆ α, we let E < F if E and F are different, and the least element of α where E and F differ belongs to F.

Let X be the set of all maps x : |P(α)| → P(α) such that the image of *x* is an ultrafilter on α. Note that every ultrafilter on α appears as image of some (actually infinitely many) elements of X .

The set X is totally ordered by setting $x < x'$ if there is an ordinal $\xi < |P(\alpha)|$ such that $x|\xi = x'|\xi$ (that is, x and x′ coincide on all the ordinals less than ξ) and $x(\xi) < x'(\xi)$ in the lexicographic order of P(α).

A Canonical Representation

Theorem. For every infinite cardinal α there is an iterated ultrapower Π_α of [0,1], definable in α, where every MV-chain of cardinality α embeds.

Corollary. For every infinite cardinal α there is an iterated ultrapower Π_α of [0,1], definable in α, such that every MV-algebra of cardinality α embeds in an algebra of functions with values in Π_α.

Thank you!