Some consequences of compactness in Łukasiewicz Predicate Logic

Luca Spada

Department of Mathematics and Computer Science
University of Salerno
www.logica.dmi.unisa.it/lucaspada

7^{th} Panhellenic Logic Symposium 2009. Patras, 16 July 2009.

Continuous t-norm based logics

Łukasiewicz logic is just one possibility in the myriad of infinite-valued generalisations of classical logic.

Among those generalisations some are meaningless, for they have very little in common with a *logic*. Yet, when one requires a few natural properties to be fulfilled, the systems arising allow deep mathematical investigations.

This is the case for **continuous t-norm based logics**. In these systems the conjunction is interpreted in an associative, commutative and weakly-increasing continuous function from $[0,1]^2$ to $[0,1]$, which behaves accordingly to classical conjunction in the limit cases 0 and 1. Such functions are **called continuous t-norm**.

Basic Logic as a common framework

As a matter of fact the most important many-valued logics studied in mathematics are based on continuous t-norms; this is the case, for instance, of Łukasiewicz logic or Gödel logic. The logical system **BL** encompasses all logics based on continuous t-norms.

The setting based on continuous t-norm, or equivalently BL, has been quite successful, for it provides a general mathematical framework for investigations on many-valued logics and offers an utter bridge towards fuzzy set theory and fuzzy logic, as t-norms are a pivotal tool in fuzzy logic.

Peculiar properties of Łukasiewicz logic

Yet Łukasiewicz logic stands out among those logics because of some of its properties. Indeed, Łukasiewicz logic is the **only** one, among continuous t-norm based logics, with a continuous implication and therefore the only logic whose whole set of formulae can be interpreted as continuous functions.

Furthermore the Łukasiewicz negation is **involutive**, namely it is such that $\neg\neg\varphi \leftrightarrow \varphi$.

Those two features, inherited from classical logic, makes Łukasiewicz logic a promising setting to test how far the methods of model theory can reach in the realm of many-valued logics.

A model theory inside many-valued logic

A model theoretic study of many-valued logic is especially important in the light of the negative results already obtained in the first order theory of these logics: the predicate version BL has a (standard) tautology problem whose complexity is not arithmetical, the same problem is Π_2-complete for Łukasiewicz logic.

Thus the favourable duality between syntax and semantics vanishes when switching to t-norm based logics and new tools must be developed.

The results so far are encouraging: recently the Robinson finite and infinite forcing were generalised to Łukasiewicz logic; here some basic results for a model theory of Łukasiewicz logic are presented and used to settle an open problem left therein.

Łukasiewicz logic

The language of the infinite-valued Łukasiewicz propositional logic, Ł, is built from a countable set of propositional variables, $Var = \{p_1, p_2, \dots, p_n, \dots\}$, and two connectives $\to$ and $\neg$.

The axioms of Ł are the following:

$$\varphi \to (\psi \to \varphi); \qquad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi));$$
$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi); \qquad (\neg\varphi \to \neg\psi) \to (\psi \to \varphi),$$

Modus ponens is the only rule of inference. The notions of proof and tautology are defined as usual.

MV-algebras

The equivalent algebraic semantics for Ł is given by the variety of MV-algebras.

An MV-algebra is a structure $\mathcal{A} = \langle A, \oplus, ^*, 0 \rangle$ such that:

- $\mathcal{A} = \langle A, \oplus, 0 \rangle$ is a commutative monoid,
- * is an involution and
- the following equations hold: $x \oplus 0^* = 0^*$ and $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$

Predicate Łukasiewicz logic

An Ł∀ language $\mathcal{L}$ is defined similarly to a language for classical first order logic, without functional symbols, taking as primitive the connectives: $\rightarrow, \neg, \forall$.

This allows the syntactical concepts of term, (atomic) formula, free or bounded variable, substitutable variable for a term, formal proof, formal theorems, etc. to be defined just as usual.

The set $V = \{x, y, z, ...\}$ is a fixed set of variables and **Form** will be used to indicate the set of formulae of $\mathcal{L}$.

Predicate Łukasiewicz logic

The axioms of Ł$\forall$ are:

(i) All the axioms of the infinite-valued propositional Łukasiewicz calculus;

(ii) $\forall x \varphi \to \varphi(t)$, where the term t is substitutable for x in φ;

(iii) $\forall x(\varphi \to \psi) \to (\varphi \to \forall x \psi)$, where x is not free in φ;

The inference rules are *Modus ponens*: from φ and $\varphi \to \psi$, derive ψ; *Generalisation*: from φ, derive $\forall x \varphi$.

Structures for predicate Łukasiewicz logic

Let $\mathcal{L}$ be a Ł∀ language with n predicate symbols and m constant symbols. Let A be an MV-algebra. An A-structure has the form

$$\mathcal{M} = \langle M, P_1^{\mathcal{M}}, ..., P_n^{\mathcal{M}}, c_1^{\mathcal{M}}, ..., c_m^{\mathcal{M}} \rangle$$

where M is a non-empty set (called the universe of the structure).

If P_i is a predicate symbol in $\mathcal{L}$ of arity k then $P_i^{\mathcal{M}}$ is a k-ary A-valued relation on A, namely a function

$$P_i^{\mathcal{M}} : M^k \rightarrow A;$$

if c_j is a constant symbol in $\mathcal{L}$ then $c_j^{\mathcal{M}}$ is an element of M.

Evaluations

Let $\mathcal{M}$ be an A-structure. An **evaluation** of $\mathcal{L}$ in $\mathcal{M}$ is a function $e : V \to M$.

Given any two evaluations e, e' of $\mathcal{L}$ and for $x \in V$ let $e \equiv_x e'$ iff $e\mid_{V\setminus\{x\}} = e'\mid_{V\setminus\{x\}}$. For any term t of $\mathcal{L}$ and any evaluation in $\mathcal{M}$ let

$$t^{\mathcal{M}}(e) = \begin{cases} e(x) & \text{if } t \text{ is a variable } x \\ c^{\mathcal{M}} & \text{if } t \text{ is a constant } c \end{cases}$$

Truth values

Given any evaluation in $\mathcal{M}$, e and any formula φ of $\mathcal{L}$, the element $\|\varphi(e)\|_{\mathcal{M}}$ of A is defined by induction, and it is called the **truth value** of φ:

$$\begin{aligned}
&\text{if } \varphi = P(t_1, ..., t_n) \text{ then} \\
&\|\varphi(e)\| = P^{\mathcal{M}}(t_1^{\mathcal{M}}(e), ..., t_n^{\mathcal{M}}(e)); \\
&\text{if } \varphi = \neg\psi \text{ then } \|\varphi(e)\| = \|\psi(e)\|^*; \\
&\text{if } \varphi = \psi \to \chi \text{ then } \|\varphi(e)\| = \|\psi(e)\| \Rightarrow \|\chi(e)\|; \\
&\text{if } \varphi = \forall x\psi \text{ then } \|\varphi(e)\| = \bigwedge\{\|\psi(e')\| \mid e' \equiv_x e\}.
\end{aligned}$$

An evaluation $e : V \to M$ is called **safe** if for any formula ψ of $\mathcal{L}$, the supremum $\bigvee\{\|\psi(e')\| \mid e' \equiv_x e\}$ exists in A (in this case the infimum $\bigwedge\{\|\psi(e')\| \mid e' \equiv_x e\}$ also exists).

If $\|\varphi\|_{\mathcal{M}}^{A} = 1$ then φ is said to be true in $\mathcal{M}$, this can be alternatively written as $\mathcal{M} \models_A \varphi$. An A-structure $\mathcal{M}$ is a **model** of a theory T if $\mathcal{M} \models_A \varphi$ for all $\varphi \in T$.

Logical consequence and satisfiability

Definition

A **standard structure** is a $[0,1]$-structure, any valuation is safe on a standard structure.
A **standard model** of a theory T is a $[0,1]$-structure which is a model of T.
A formula φ is called ***A*-logical consequence** of a theory T, in symbols $T \models_A \varphi$, if every A-model of T is also an A-model of φ. In particular, when this is true for standard models then I write $T \models_{[0,1]} \varphi$ or $T \models \varphi$.

Definition

A formula φ is **generally satisfiable** if there exists a model $\mathcal{M}$ such that $\|\varphi\|_{\mathcal{M}} = 1$. If the model can be taken standard then φ is called just **satisfiable**. The previous definitions naturally generalise to theories. A theory T is **consistent** if $T \not\models \bot$.

Weak completeness and compactness

All the main results in this talk hinge on the following theorems.

Theorem (Weak Completeness (Belluce and Chang 1963))

Any consistent theory T of Ł$\forall$ has a standard model.

Theorem (Compactness)

Let T be a theory in Ł$\forall$:

(i) *If T is finitely generally satisfiable then T is generally satisfiable.*

(ii) *If T is finitely satisfiable then T is satisfiable.*

(iii) *If for any MV-algebra A, $T \models_A \varphi$ then there exists a finite $T_0 \subseteq T$ such that for any MV-algebra A $T_0 \models_A \varphi$*

(iv) *If $T \models_{[0,1]} \varphi$ then* **in general it is false** *that there exists a finite $T_0 \subseteq T$ such that $T_0 \models_{[0,1]} \varphi$.*

A hierarchy on formulae

Henceforth $\mathcal{L}$ is assumed to be a fixed language of Ł$\forall$ and all structures are standard.

Definition

A formula of $\mathcal{L}$ belongs to the set Σ_n (Π_n, respectively) if it is equivalent to a formula with *n* blocks of quantifier, where each block is either empty or constituted of an uninterrupted sequence of the same quantifier, $\exists$ or $\forall$, and the first block is made of $\exists$'s ($\forall$'s respectively).

As in the classical case one has $\Sigma_n \cup \Pi_n \subseteq \Sigma_{n+1} \cap \Pi_{n+1}$.

Relations among models

Let $\mathcal{M}$ be an structure, $\mathcal{L}(\mathcal{M})$ is the expansion of the language $\mathcal{L}$ with a new constant symbol for each element of M.

The **diagram** of $\mathcal{M}$, i.e. the set of atomic formulae φ in $\mathcal{L}(\mathcal{M})$ such that $\|\varphi\|_{\mathcal{M}} = 1$, is indicated by $D(\mathcal{M})$; Th$(\mathcal{M})$ is the set of formulae φ such that $\|\varphi\|_{\mathcal{M}} = 1$.

Definition

If $\mathcal{M}_1 \subseteq \mathcal{M}_2$ are two structures and for any $\varphi \in D(\mathcal{M}_1)$, $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_2 \models \varphi$ then $\mathcal{M}_1$ is a **substructure of** $\mathcal{M}_2$, in symbols $\mathcal{M}_1 \leq \mathcal{M}_2$. If the same is true for *any* sentence of $\mathcal{L}(\mathcal{M}_1)$ than $\mathcal{M}_1$ is an **elementary substructure of** $\mathcal{M}_2$, written $\mathcal{M}_1 \preceq \mathcal{M}_2$

Łoś-Tarski Theorem for Łukasiewicz logic

Proposition

Let T be a theory, let $T_\forall$ be the set of logical consequences of T which are in Π_1 and let $\mathbf{K}$ be the class of all substructures of models of T. Then $\mathbf{K}$ is the class of models of $T_\forall$.

Proof.

$\mathcal{M} \in \mathbf{K}$ then $\mathcal{M} \models T_\forall$ is straightforward. Let $\mathcal{M} \models T_\forall$, then $D_\forall(\mathcal{M}) \cup T$ is finitely satisfiable (if it were not then $\bigwedge \Psi \models_{[0,1]} \neg \bigwedge \Phi$, but $\neg \bigwedge \Phi \in \Pi_1$ $\lightning$.)
So there exists $\mathcal{N} \models D_\forall(\mathcal{M}) \cup T$ whence $\mathcal{M} \hookrightarrow \mathcal{N}$ and $\mathcal{M} \in \mathbf{K}$ □

Corollary (Łoś-Tarski Theorem for Łukasiewicz logic)

A theory is preserved under substructure if, and only if, it is equivalent to a universal (i.e. Π_1) theory.

(Elementary) chains

Definition

Let α be an ordinal and $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ a family of $\mathcal{L}$-structure. The structures $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ are a **chain** if for any $\lambda_1 \leq \lambda_2 < \alpha$, $\mathcal{M}_{\lambda_1} \leq \mathcal{M}_{\lambda_2}$.
If for any $\lambda_1 \leq \lambda_2 < \alpha$, $\mathcal{M}_{\lambda_1} \preceq \mathcal{M}_{\lambda_2}$ then $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ is called **elementary chain**.

Lemma

Let $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ be an elementary chain. Then for every $\lambda \in \alpha$, $\mathcal{M}_\lambda \preceq \bigcup_{\lambda\in\alpha} \mathcal{M}_\lambda$

T is an **inductive** theory if it is closed under unions of chains.

Theorem (Chang-Łoś-Suszko Theorem for Łukasiewicz logic)

A theory is inductive if, and only if, it is equivalent to a Π_2 theory.

Proof.

If $T \in \Pi_2$ then it is straightforward to prove that T is inductive.
Let T be inductive. If $\mathcal{M} \models T_{\forall_2}$ then $T \cup \mathrm{Th}_\exists(\mathcal{M})$ is finitely satisfiable (if not $\bigwedge \Phi \models_{[0,1]} \neg \bigwedge \Psi$, but then $\neg \bigwedge \Psi \in T_\forall$ $\lightning$.)
So there exists $\mathcal{N} \models T \cup \mathrm{Th}_\exists(\mathcal{M})$ s.t. $\mathcal{M} \hookrightarrow \mathcal{N}$.
Every existential sentence of $L(\mathcal{M})$ which is true in $\mathcal{N}$ holds in $\mathcal{M}$, hence $D(\mathcal{N}) \cup \mathrm{Th}(\mathcal{M})$ is satisfiable, so it has a model $\mathcal{M}_1$ which is an extension of $\mathcal{N}$ and an elementary extension of $\mathcal{M}$.

$$\mathcal{M} \leq \mathcal{N} \leq \mathcal{M}_1 \leq \mathcal{N}_1 \leq \ldots$$

Let $\mathcal{O}$ be the limit of this chain. $\mathcal{O} \models T$, for T is inductive; furthermore $\mathcal{O}$ is an elementary extension of $\mathcal{M}$, because the chain $\{\mathcal{M}_i\}_{i\in\omega}$ is elementary. Therefore $\mathcal{M} \models T$. □

Model companions

The above characterisation is extremely useful, when dealing with model complete theories.

Corollary

When the model companion of a theory is axiomatisable, it is equivalent to a $\forall\exists$ *theory.*

Proof.

In a model companion every chain is elementary. □

From this it is also easy to see that

Corollary

There exists at most one model companion of a theory.

Generic models

Recently the notion of model theoretic forcing was extended to Łukasiewicz logic, leading to the study of the class of *generic models*, $\mathfrak{G}_{\mathbf{K}}$, contained in a given class $\mathbf{K}$.

The class $\mathfrak{G}_{\mathbf{K}}$ was proved to contain the subclass of existentially closed models of $\mathbf{K}$. The Chang-Łoś-Suszko theorem for Łukasiewicz logic enables to complete this result.

Proposition

Given a theory T, if $\mathfrak{G}_{\mathrm{Mod}(T)}$ is axiomatisable then **it is** *the class of existentially closed models of T.*

Proof.

Let $\mathcal{M}$ be a existentially closed model of T, then it embeds in a model $\mathcal{N} \in \mathfrak{G}_{\mathrm{Mod}(T)}$. The class $\mathfrak{G}_{\mathrm{Mod}(T)}$ is inductive, so if it is axiomatisable then it is equivalent to a Π_2 theory. Since $\mathcal{M}$ is existentially closed, it is easy to see that it satisfies the same Π_2 formulae of $\mathcal{N}$,whence $\mathcal{M} \in \mathfrak{G}_{\mathrm{Mod}(T)}$. □

Further reading

Belluce, L.P., Chang, C.C.:
A weak completeness theorem for infinite valued first-order logic.
Journal of Symbolic Logic **28**(1) (1963) 43–50

Chang, C. C.:
Omitting types of prenex formulas,
Journal of Symbolic Logic **32** (1967) 61-74

Cignoli, R., D'Ottaviano, I., Mundici, D.:
Algebraic Foundations of Many-valued Reasoning. Volume 7 of Trends in Logic, Studia Logica Library.
Kluwer Academic (2000)

Di Nola, A., Georgescu, G., Spada, L.:
Forcing in Łukasiewicz predicate logic.
Studia Logica **89**(1) (May 2008) 111–145

Hájek, P.:
Metamathematics of Fuzzy Logic. Volume 4 of Trends in Logic, Studia Logica Library.
Kluwer Academic, Berlin (1998)

Robinson, A.:
Forcing in model theory.
In: Actes du Congrès International des Mathématiciens, Nice. Volume 1., Paris, Gauthier-Villars (1970) 145–150