Geometric aspects of MV-algebras

Luca Spada
Università di Salerno

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Contents

- Crash tutorial on MV-algebras.
- Dualities for semisimple MV-algebras.
- Non semisimple MV-algebras.
- Ind and pro completions with an application to MV-algebras.

This work is based on results obtained with

L. Cabrer and V. Marra.

Lukasiewicz logic

It is a logic L in which the formulas may take **any truth value in the real interval** [0,1].

L can defined in terms of → as **the only one** such that

- It is closed under Modus Ponens.
- The connective → is **continuous**.
- The **order of premises** is irrelevant.
- For any truth-values $x, y \in [0,1]$,

$x \to y$ **equals 1 precisely** when $x \leq y$.

MV-algebras

An MV-algebra is a structure $(A,\oplus,\neg,0)$ such that $(A,\oplus,0)$ is a **commutative monoid** and the following axioms hold:

1. $\neg 0 \oplus x = \neg 0$
2. $\neg \neg x = x$
3. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

Any MV-algebra has a **lattice structure** given by setting

$$\neg(\neg x \oplus y) \oplus y = x \vee y$$

Examples of MV-algebras

1. Any Boolean algebra is an MV-algebra where $\oplus$ satisfies $x \oplus x = x$.

2. Consider [0,1] with the operations:

$$x \oplus y := \min\{x+y, 1\} \quad \text{and} \quad \neg x := 1-x$$

(Example: $0.3 \oplus 0.2 = 0.5$ but $0.7 \oplus 0.8 = 1$)

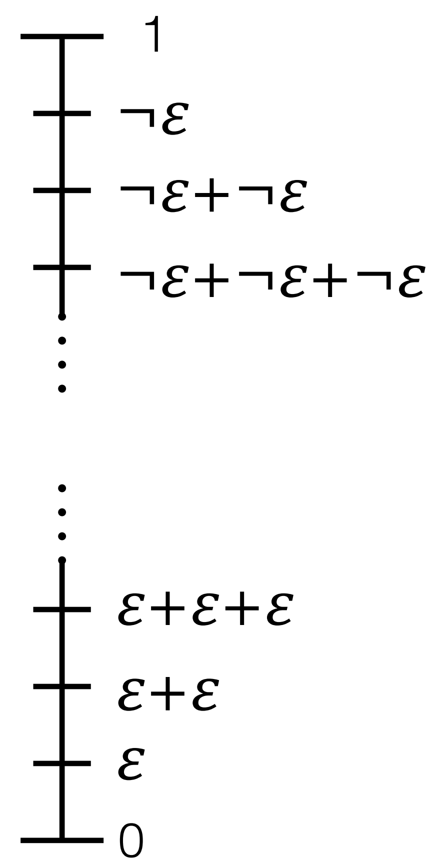
$([0,1], \oplus, \neg, 0)$ is an MV-algebra.

Theorem 1: The algebra [0,1] generates the variety of MV-algebras.

Examples of MV-algebras

3. Let ε be just a symbol, consider $\{n\varepsilon, n(\neg\varepsilon) \mid n\in\mathbb{N}\}$ endowed with the operations:

- $n\varepsilon \oplus m\varepsilon := (n+m)\varepsilon$
- $n(\neg\varepsilon) \oplus m(\neg\varepsilon) := 1$
- $n\varepsilon \oplus m(\neg\varepsilon) := (n-m)\neg\varepsilon$
- $\neg n(\neg\varepsilon) := (n)\varepsilon$
- $\neg n(\varepsilon) := (n)\neg\varepsilon$



This is called the **Chang's algebra**. It is **not semisimple**.

Simple and semisimple MV-algebras

Simple MV-algebra = only trivial congruences
= subalgebra of [0,1] = $\mathbb{IS}([0,1])$

Theorem 2: An MV-algebra is **simple** if, and only if, it is a **subalgebra** of [0,1].

Semisimple MV-algebra = subdirect product of simple algebras = $\mathbb{ISP}([0,1])$.

Ideals

If A is an MV-algebra, a non empty $P \subseteq A$ is called **ideal** if

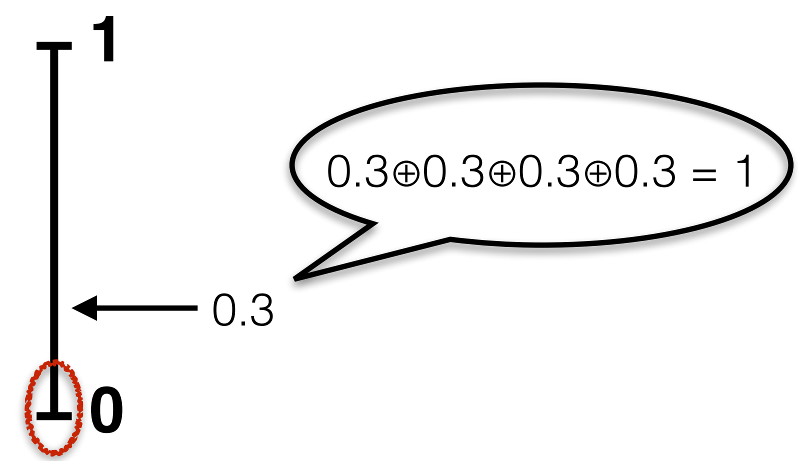
- P is downward closed,
- $a,b \in P$ implies $a \vee b \in P$,
- $a,b \in P$ implies $a \oplus b \in P$.

P is called **maximal** if it is maximal among the proper ideals w.r.t. the inclusion order.

Our examples

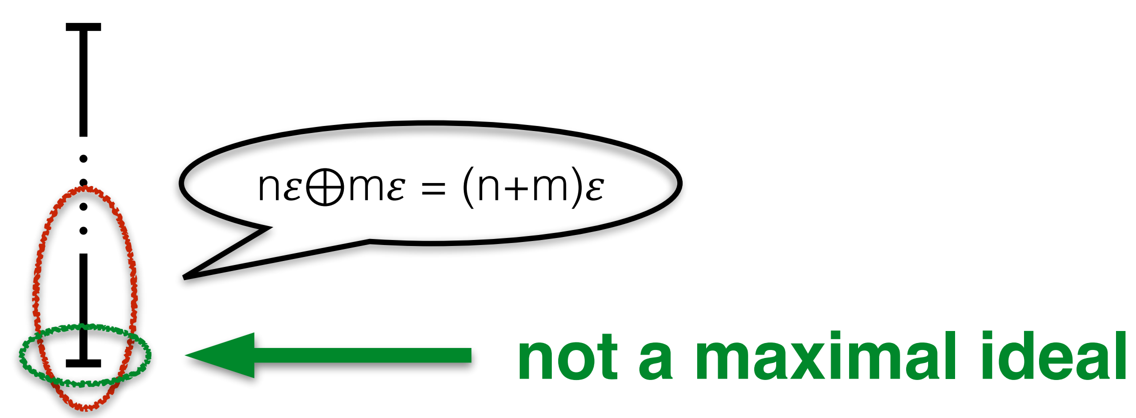
The algebra {0,1}

[0,1]



They are indistinguishable by simply using their maximal ideals.

Chang's algebra



Two different things

These are two **different phenomena**, and it is important to keep them distinct.

To begin with let us concentrate only on **semisimple** MV-algebras.

Part I:
Semisimple MV-algebras

Finite MV-algebras

- Finite MV-algebras are products of finite linearly ordered MV-algebras.

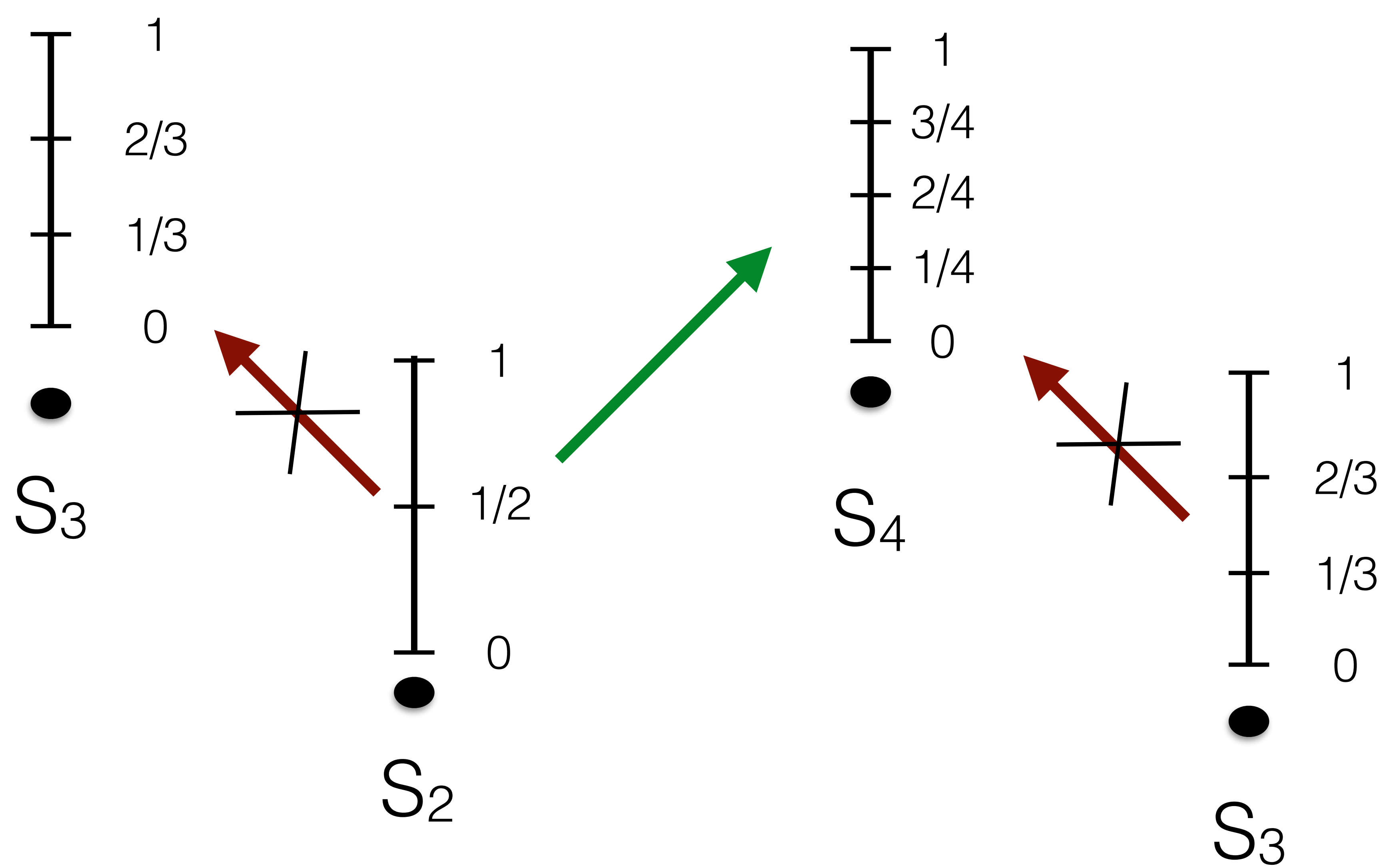
- All finite linearly ordered MV-algebras are simple.

$$\mathbf{S_n = \{0, 1/n, 2/n, \ldots, (n\text{-}1)/n, 1\}}$$

with operations inherited from the MV-algebra [0,1].

- $S_2 = \{0, 1/2, 1\}$, $S_3 = \{0, 1/3, 2/3, 1\}$, etc.

The duals of finite MV-algebras



The algebras can be reconstructed if we attach natural numbers to points.

A duality for "finitely valued" MV-algebras.

Niederkorn (2001) using the theory of *Natural Dualities* proves that

$\mathbb{ISP}(S_n)$ **is dual to** $(X, D_1, \ldots, D_n)$

X: Stone space

$D_1, \ldots, D_n$: unary predicates [...]

A duality for locally finite MV-algebras.

Cignoli-Dubuc-Mundici (2004), using ind- and pro-completions, prove that

Locally finite MV-algebras **are dual to** $(X, f\colon X \to s\mathbb{N})$

X: Stone space

f: continuous map into the "super natural numbers"

A further extension

However, the situation is more complex, indeed:

Theorem 3. Every compact Hausdorff space is homeomorphic to Max(A), for some MV-algebra A.

- How can we attach natural numbers to the points of an abstract compact Hausdorff space?

- How can we use those numbers to recover the structure of the MV-algebra?

The following definition is CRUCIAL.

$\mathbb{Z}$-maps

Let C, D be sets. A continuous map

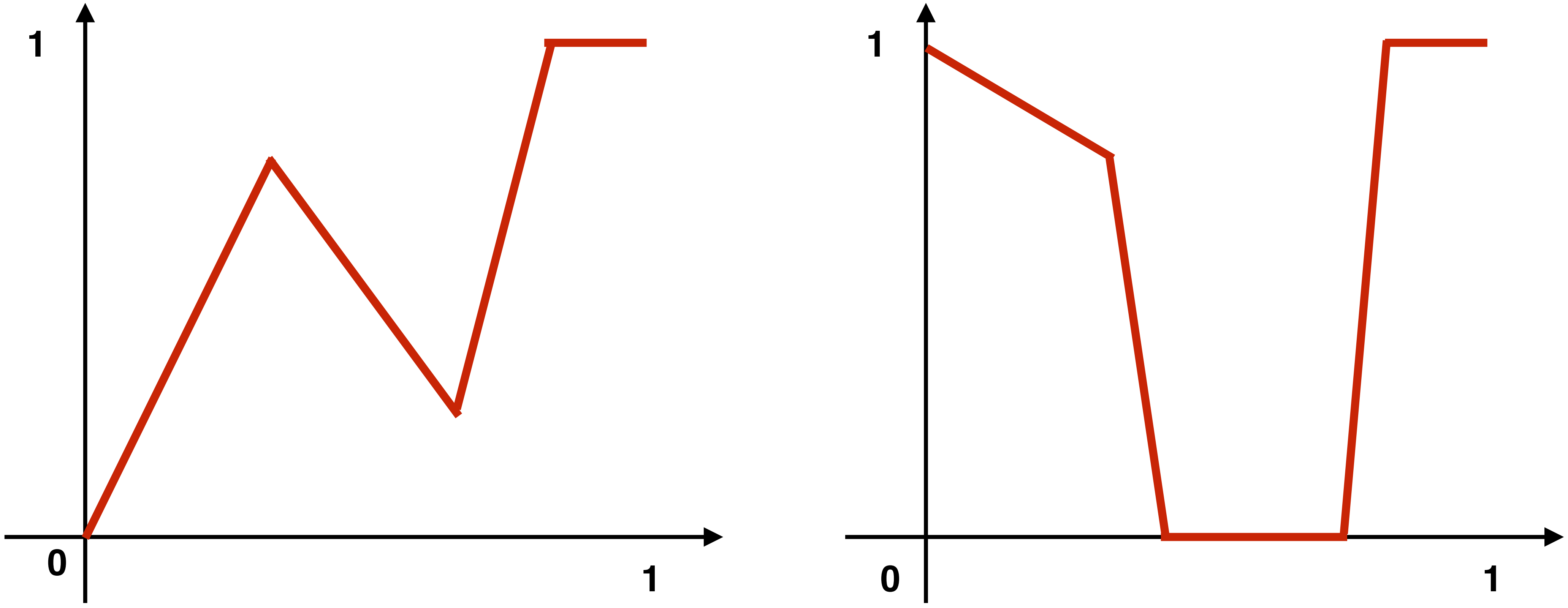
$$z = (z_{d \in D}) : [0,1]^C \longrightarrow [0,1]^D$$

is called a $\mathbb{Z}$-map if for each $d \in D$, z_d is **piecewise linear** with **integer coefficients**.

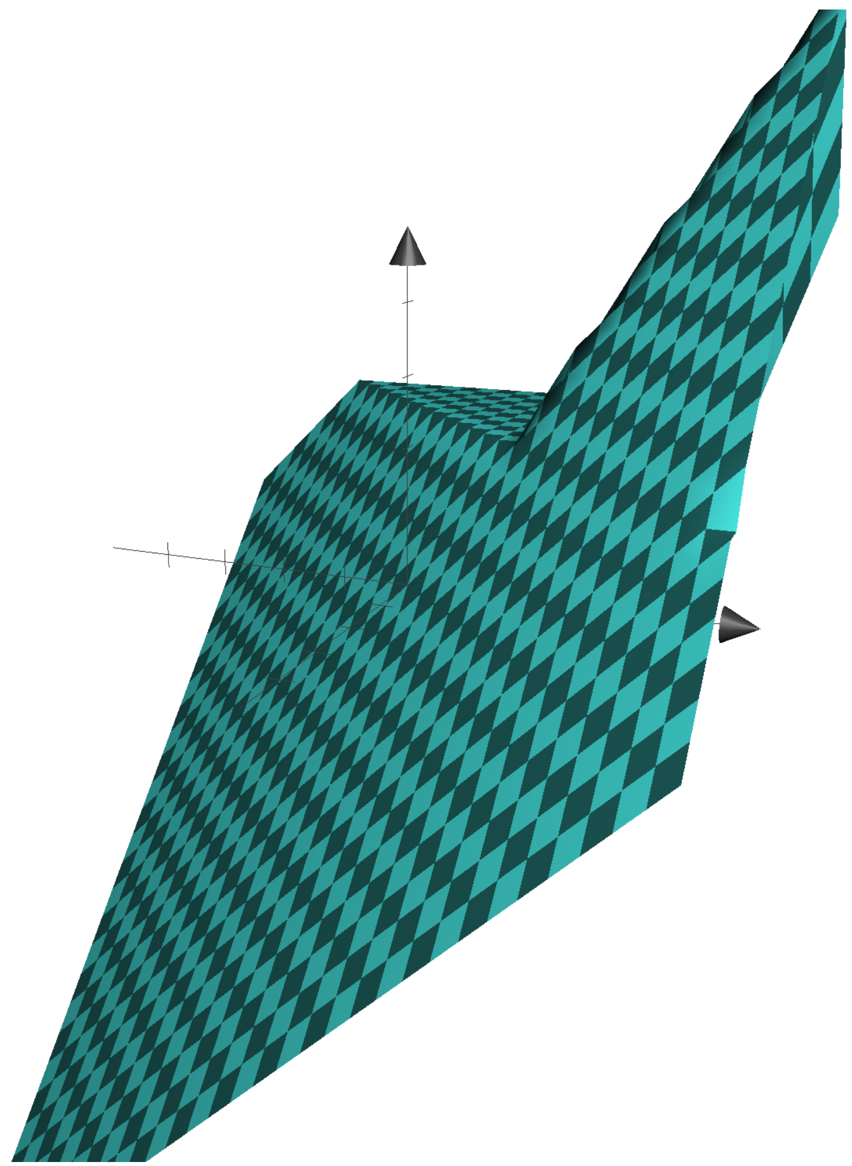
If $P \subseteq [0,1]^C$ and $Q \subseteq [0,1]^D$, a $\mathbb{Z}$-map $z : P \to Q$ is a simply a **restriction** of $\mathbb{Z}$-map from $[0,1]^C$ into $[0,1]^D$.

Let **T** be the **category of subspaces** of $[0,1]^C$, for any set C, and $\mathbb{Z}$-maps among them.

$\mathbb{Z}$-maps from [0,1] into [0,1]

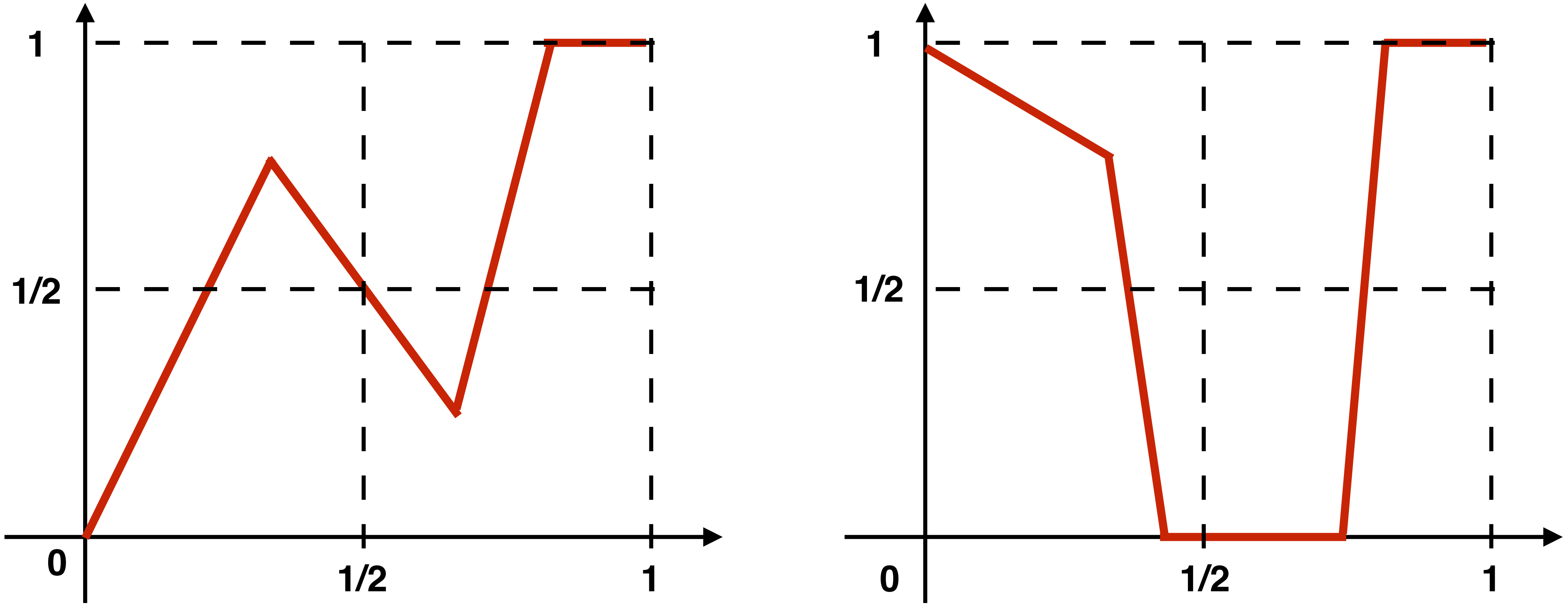


A $\mathbb{Z}$-map from $[0,1]^2$ to $[0,1]$



$\mathbb{Z}$-maps

$\mathbb{Z}$-maps have interesting properties, e.g., they respect denominators.



$\{n \cdot \mathbf{1/2} \mid n \in \mathbb{N}\} = \{0, 1/2, 1\}$

$\{n \cdot \mathbf{1/6} \mid n \in \mathbb{N}\} = \{0, 1/6, 1/3, 1/2, 2/3, 5/6, 1\}$

McNaughton theorem

Theorem 4 (McNaughton)
The MV-terms in *n* variables interpreted on the MV-algebra $[0,1]^n$ are exactly the $\mathbb{Z}$-maps from $[0,1]^n$ into $[0,1]$.

Corollary (to be used later)
The free *n*-generated MV-algebra is isomorphic to the algebra of $\mathbb{Z}$-maps from $[0,1]^n$ into $[0,1]$.

The framework of natural dualities

- On the one hand we have $\mathbf{MV_{ss}} = \mathbb{ISP}([0,1])$,
- On the other hand, $\mathbf{T} = \mathbb{IS}_{\mathbf{c}}\mathbb{P}([0,1])$.
- In fact, [0,1] plays both the role of an MV-algebra and of an element of **T**.
- The functors $\hom_{\mathbf{T}}(\,—\,, [0,1])$ and $\hom_{\mathbf{MV}}(\,—, [0,1])$ **form a contravariant adjunction.**

The (contravariant) hom functors

hom_{MV}(A, [0,1]) is bijective to **Max(A)**, and since

$$\mathrm{hom}_{\mathbf{MV}}(A, [0,1]) \subseteq [0,1]^A$$

it inherits the product topology.

Do not forget it!

The space **hom_{MV}(A, [0,1])** with the **product topology** is homeomorphic to **Max(A)** with the **Zariski topology**.

hom_{T}(X, [0,1]) has MV-operations defined point-wise.
I will often write $\mathbb{Z}(X)$ for $\mathrm{hom}_{\mathbf{T}}$(X, [0,1]).

A representation as algebras of $\mathbb{Z}$-maps

For any semisimple MV-algebra,

$$A \cong \mathbb{Z}(\mathrm{Max}(A))$$

For any closed $X \subseteq [0,1]^C$,

$$X \cong_{\mathbb{Z}} \mathrm{Max}(\mathbb{Z}(X))$$

Theorem 5
The category of semisimple MV-algebras with their homomorphisms

is dually equivalent

to the category of **closed subspaces of $[0,1]^A$**, with *A* any set, and $\mathbb{Z}$-maps as arrows.

Finitely presented MV-algebras

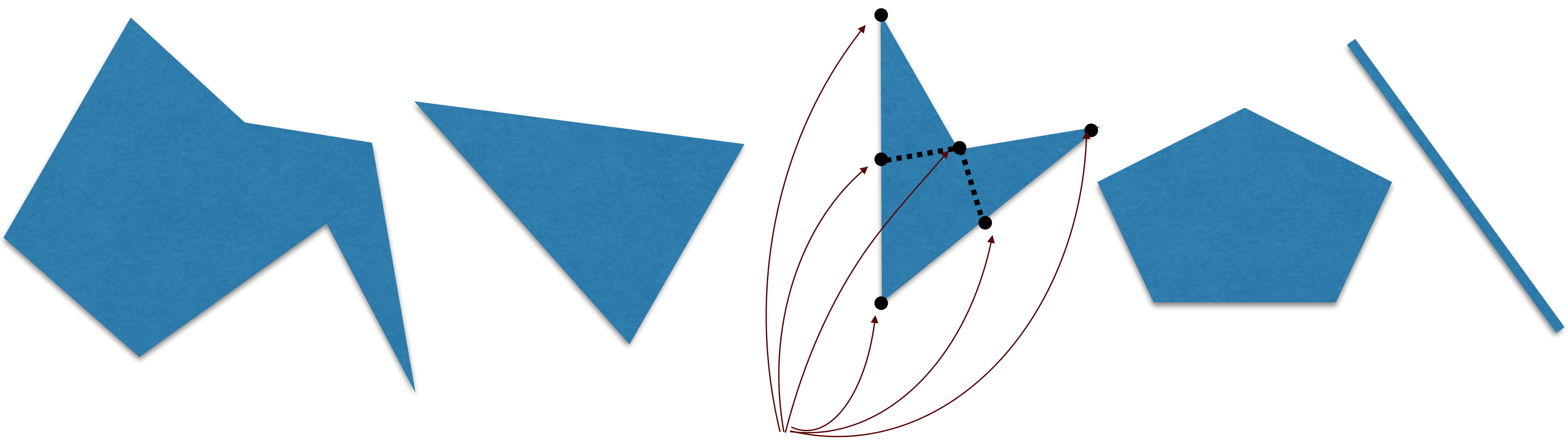
A finitely presented algebra is the quotient of a finitely generated free algebra over a finitely generated ideal.

$$\frac{Free(x_{1,\ldots,}x_n)}{< f\,(x_1,\ldots,x_n) >_{\mathrm{id}}}$$

The equation $f\,(x_1,\ldots,x_n) = 0$ defines a closed subspace of $[0,1]^n$

Rational polyhedra

In the case of MV-algebras, those equations define a **rational polyhedron.**



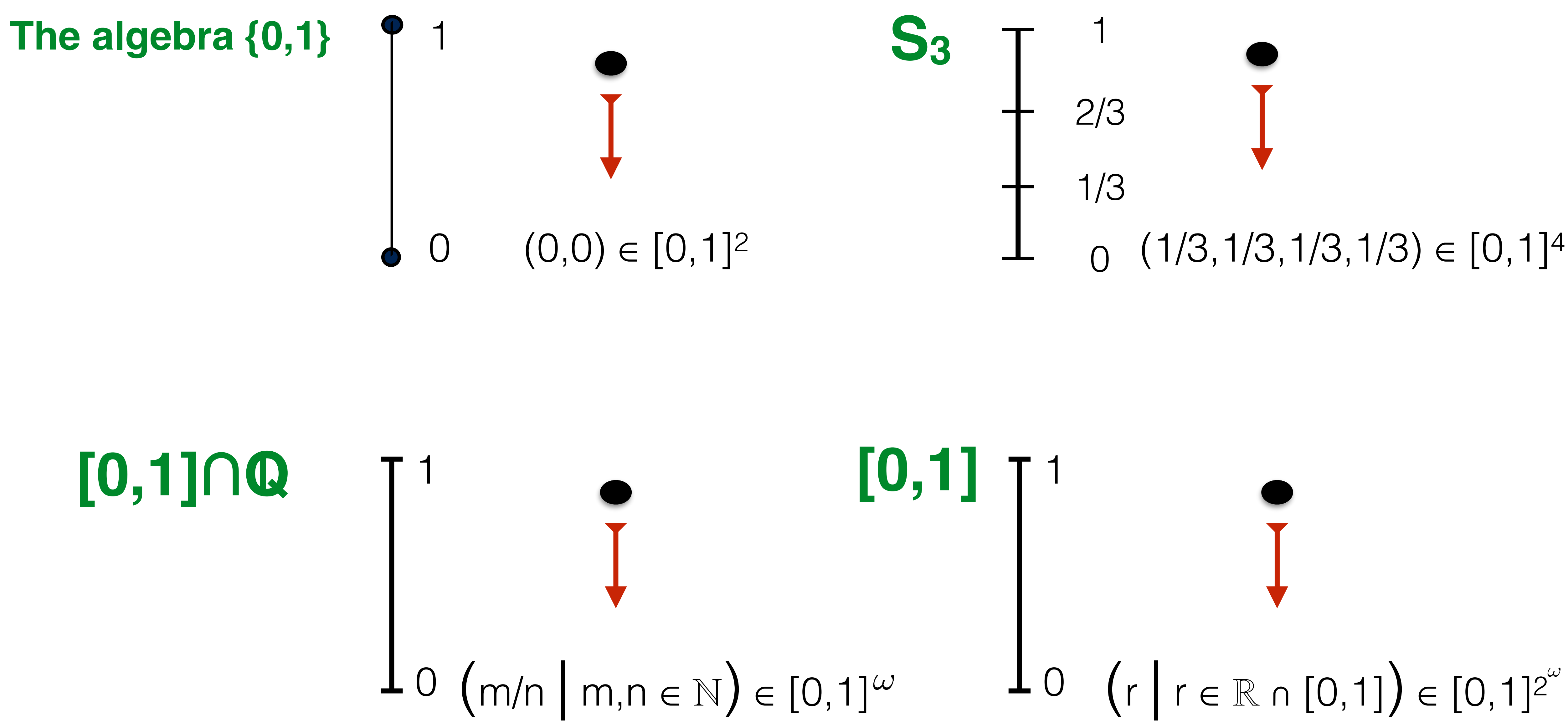
Points with rational coordinates

More precisely, a rational polyhedron is a finite union of convex hulls of rational points in $[0,1]^n$.

The duality for finitely presented MV-algebras

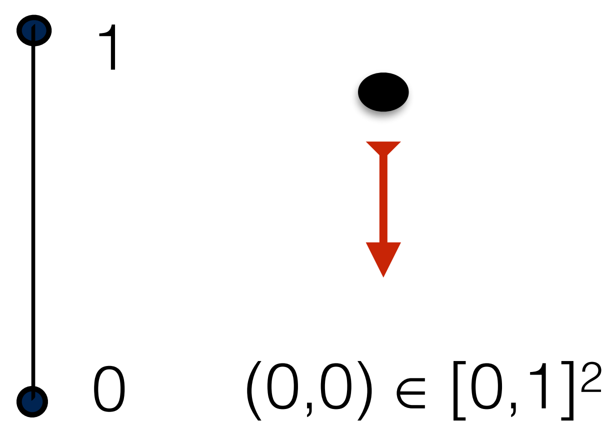
Corollary (sort of)
The category of **finitely presented MV-algebras** with their homomorphisms
is dually equivalent
to the category $\mathbf{P_Z}$ of **rational polyhedra and $\mathbb{Z}$-maps**.

Our examples

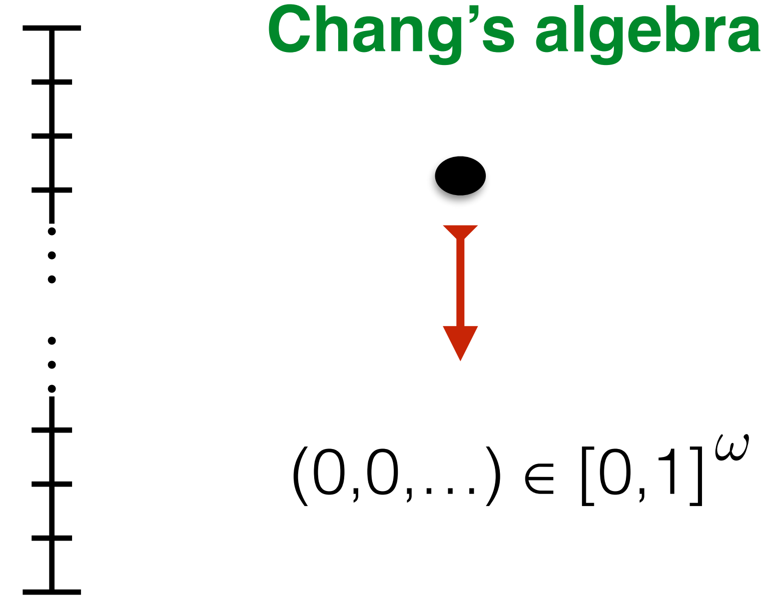


But…

The algebra {0,1}



Chang's algebra

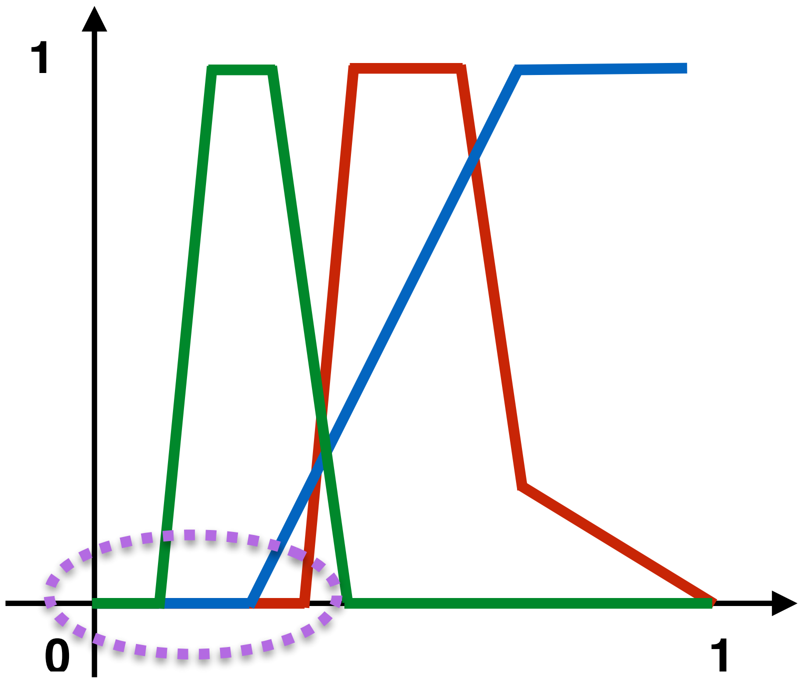


Are still indistinguishable!

Part 2:
Non semisimple

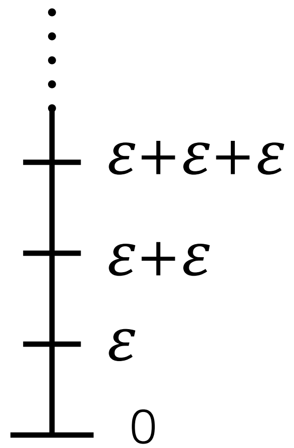
Chang's algebra revisited

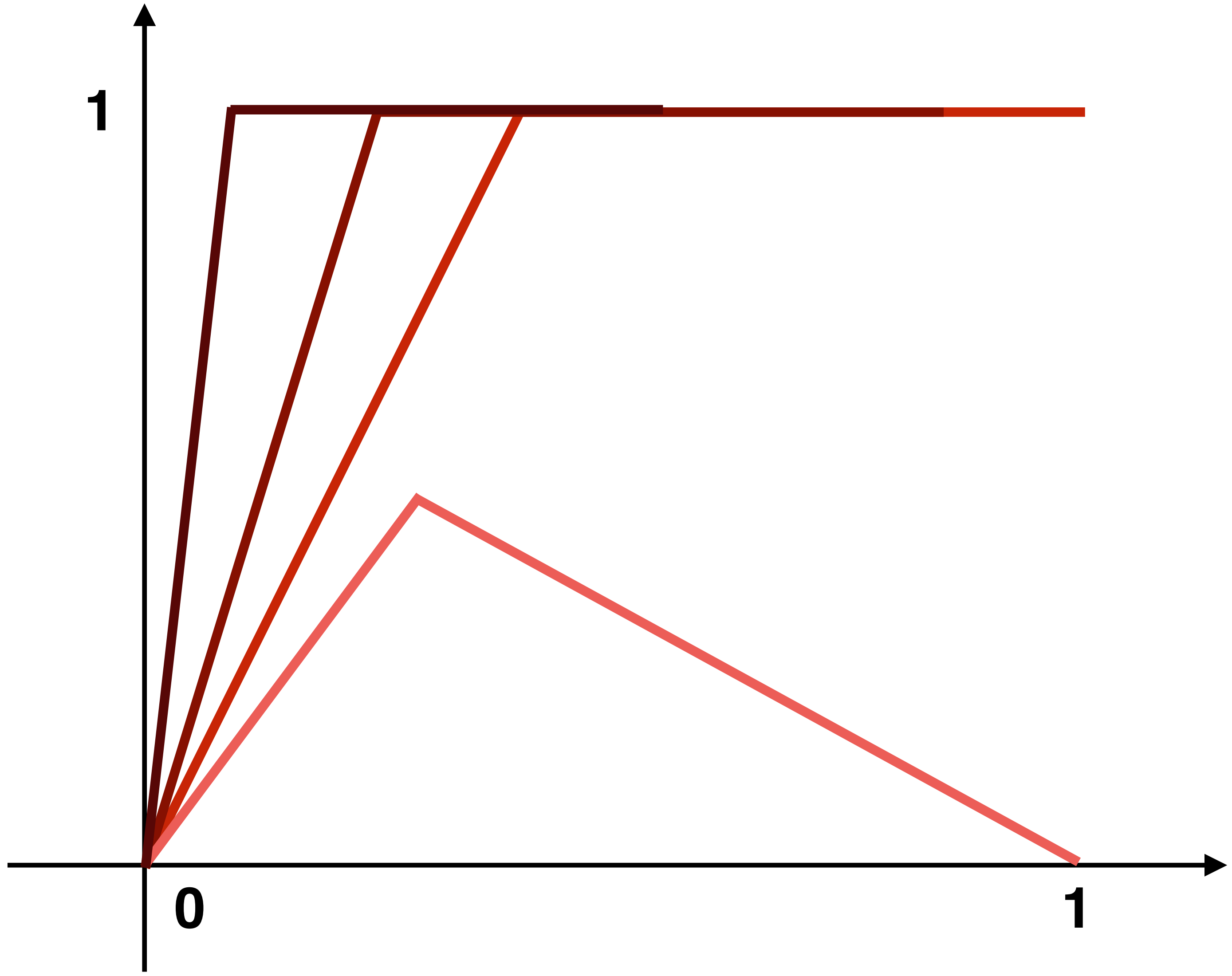
Consider the set of $\mathbb{Z}$-maps from [0,1] into [0,1] for which there exists a neighbour of the point 0, in which they vanish.



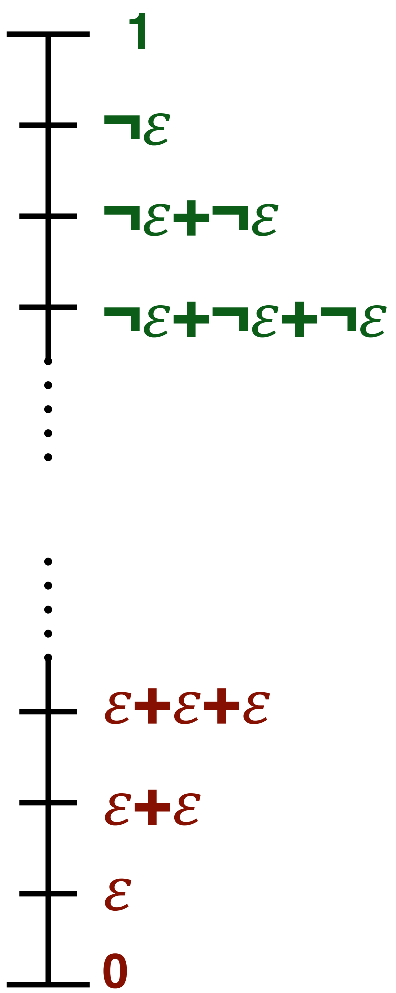
Take the quotient of Free(1), by this ideal.

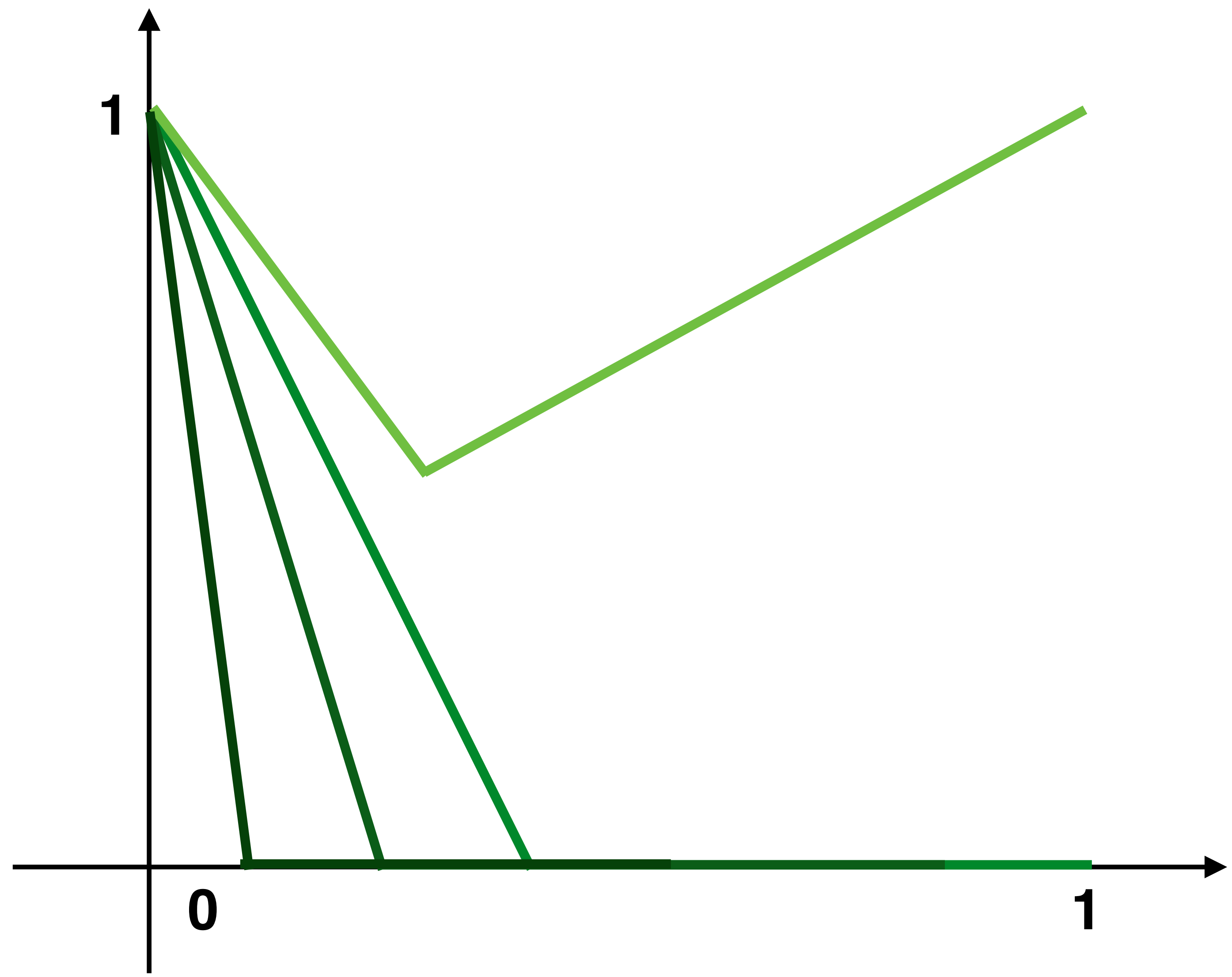
Chang's algebra





Chang's algebra

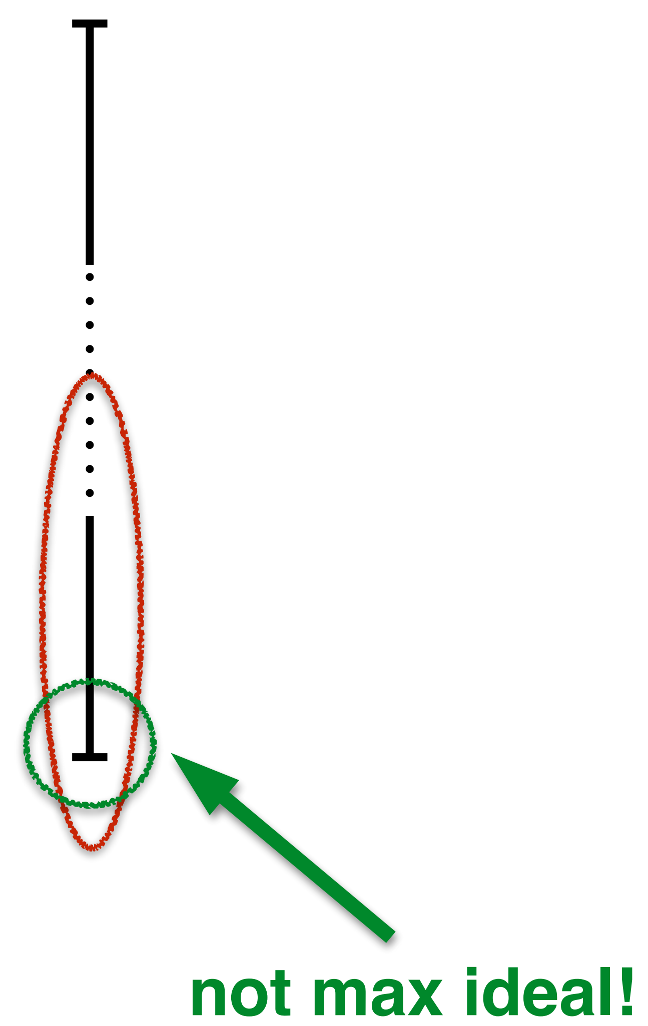




Max(A) and Spec(A)

- **Maximal ideals** correspond to **points** in the dual.
- **Prime ideals** correspond to some sort of **neighbourhood systems** of the maximal ideal that contains them.

Chang's algebra



How can we concretely describe this additional piece of information?

Johnstone's approach

In *Stone Spaces* P. Johnstone uses **ind-** and **pro-completions** to prove some classical dualities..

- Start with the duality between **finite sets** and **finite Boolean algebras**. Take all directed limits in the first case and all directed colimits in the second case….
- Start with **Birkhoff's duality** between **finite distributive lattices** and **finite posets**. Take again (directed) limits and colimits….

Ind- and pro- completions

- The **ind-completion** of a category C is a new category whose objects are **directed diagrams in C**.

- Arrows in **ind-C** are **families of equivalence classes** of arrows in C. (We'll get back to this later.)

- The **pro-completion** is formed similarly.

Corollary

If **A** is a finitary algebraic category, then there is an equivalence

Ind-$\mathbf{A}_{fp} \simeq \mathbf{A}$.

An application to MV

Let B and C be two categories,

if $B \simeq C$ then ind-$B \simeq$ (pro-$C^{op})^{op}$.

Now, $\mathbf{MV_{fp}} \simeq \mathbf{(P_Z)^{op}}$, so

$\mathbf{MV} \simeq$ ind-$MV_{fp} \simeq ((\text{pro-}(P_Z)^{op})^{op})^{op} = \mathbf{(pro\text{-}P_Z)^{op}}$.

Theorem 6: $\mathbf{MV \simeq (pro\text{-}P_Z)^{op}}$

MV-algebras (general case)

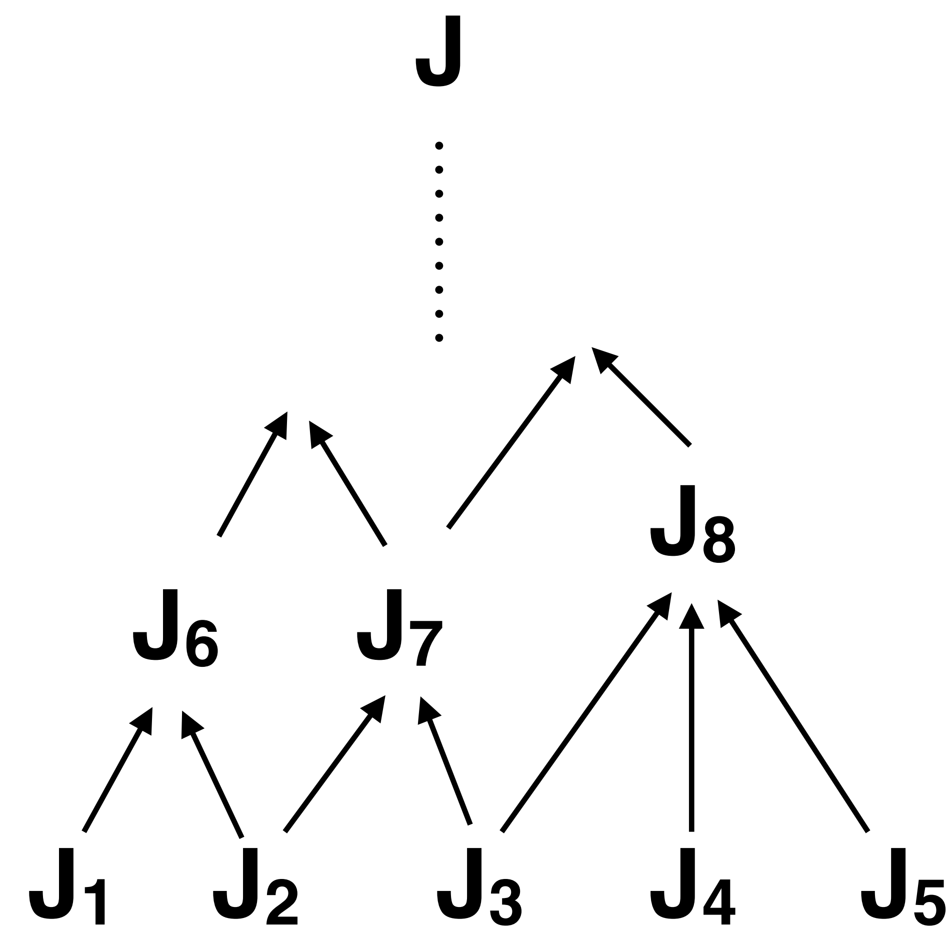
Any algebra is the quotient of a free algebra over some ideal.

$$\frac{\mathit{Free}(S)}{J}$$

Finitely presented algebras as building blocks

Start with any algebra $\frac{\textbf{\textit{Free}}\textbf{(S)}}{\textbf{J}}$

One can form a directed diagram by taking all **finite subsets** of **J**

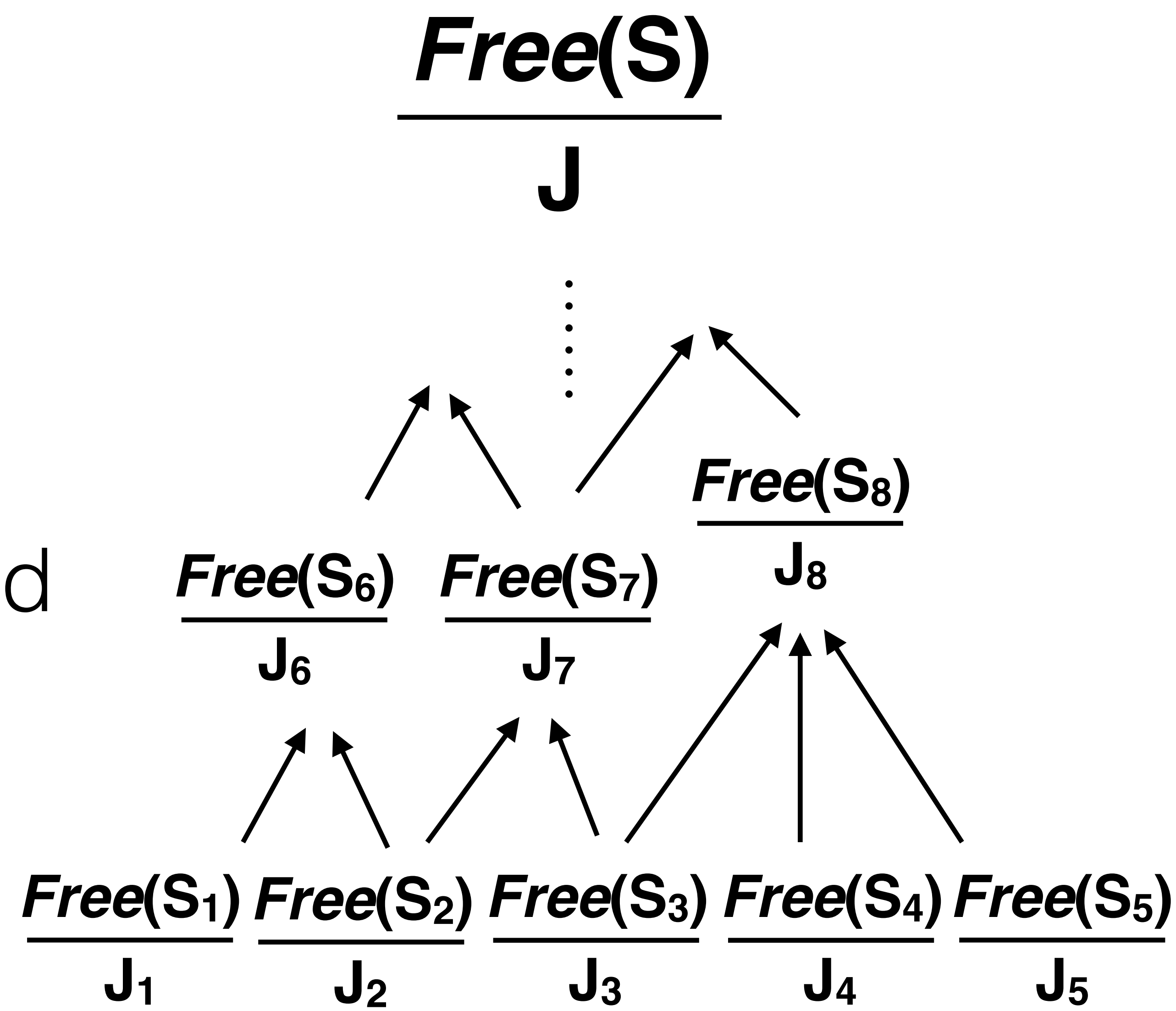


Finitely presented algebras as building blocks

Start with any algebra $\frac{\textbf{\textit{Free}}\textbf{(S)}}{\textbf{J}}$

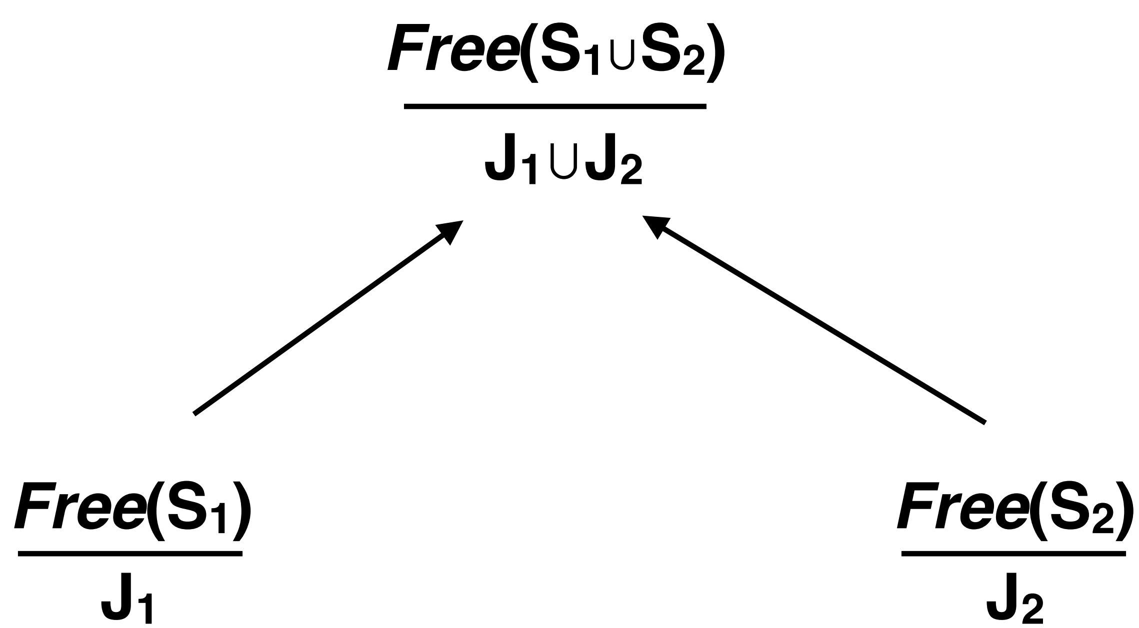
One can form a directed diagram by taking all **finite subsets** of **J**

This corresponds to a directed diagram of algebras

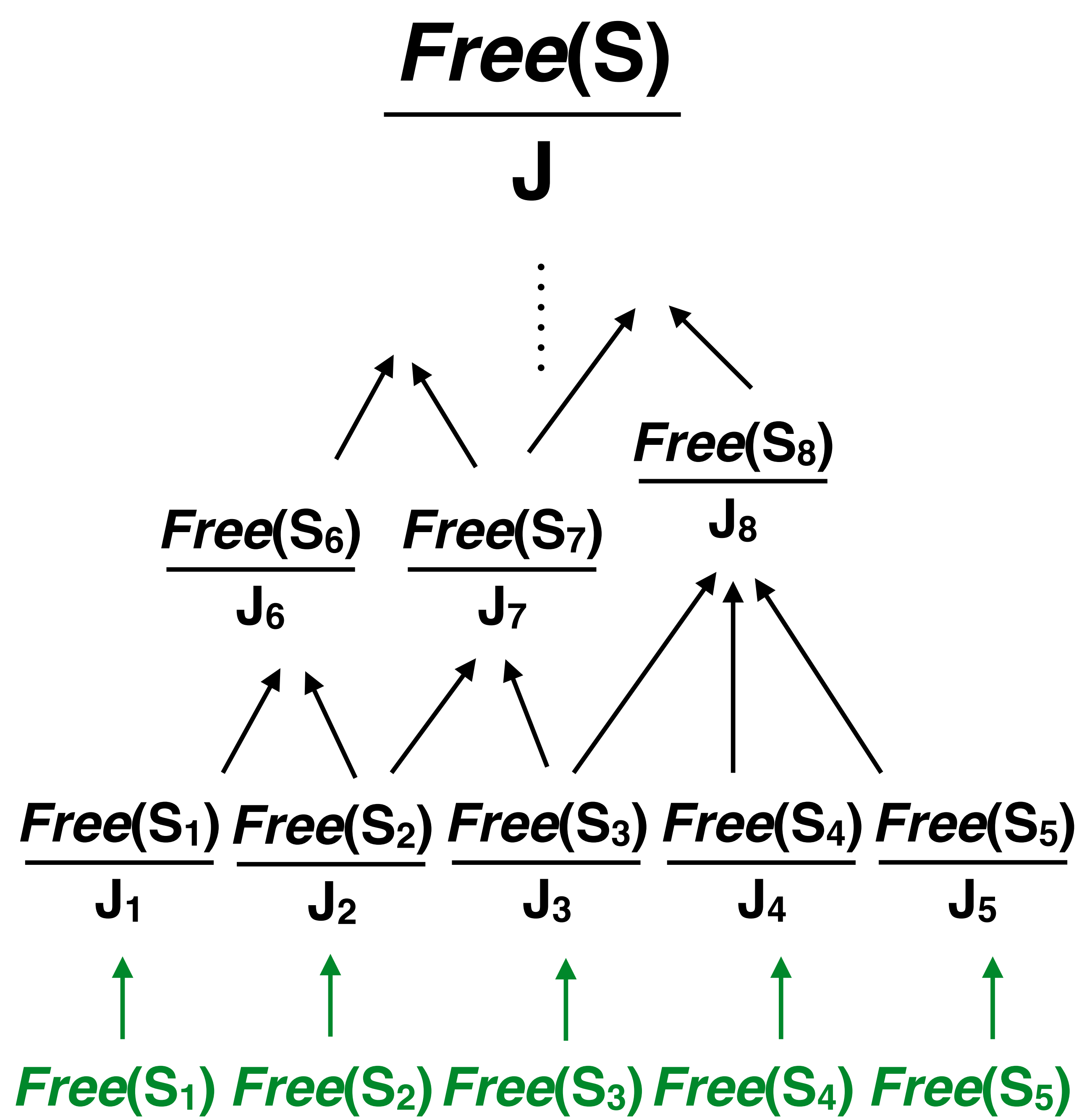


Directness of the diagram

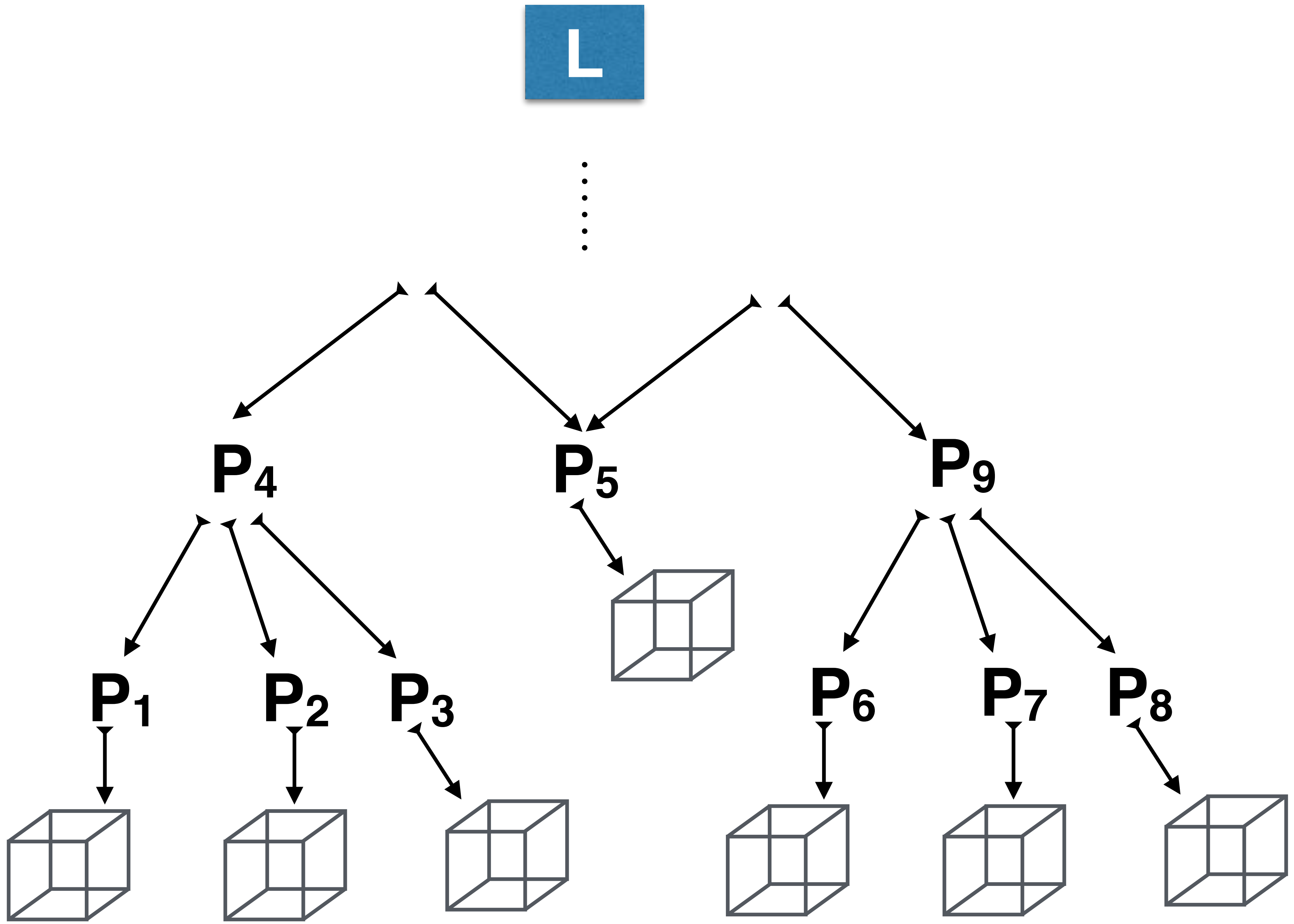
It is clear that the diagram is directed



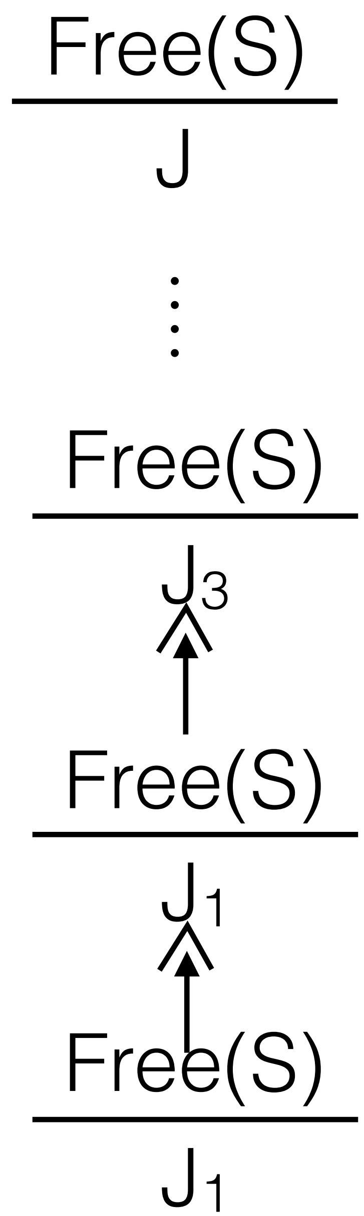
Finitely presented algebras as building blocks



Limits of rational polyhedra

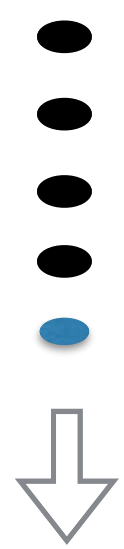


Finitely generated MV-algebras

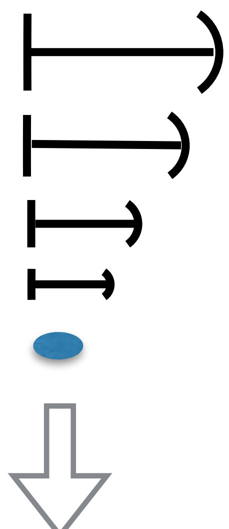


For **finitely generated** MV-algebra, it is enough to consider diagrams that have the order type of ω

Recognising Chang



$[0,1]^\omega$

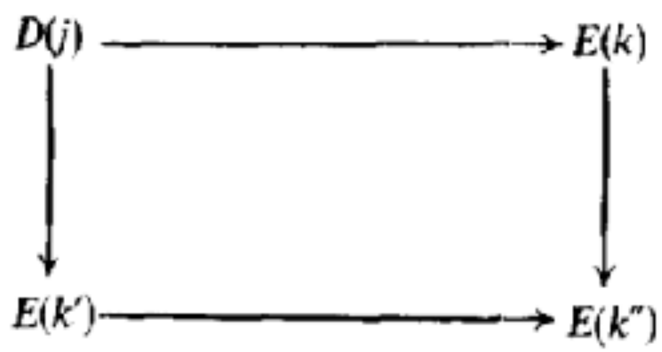


$[0,1]^\omega$

Stone spaces, pag. 225

$$\hom(D : \mathbf{J} \to \mathbf{C}, E : \mathbf{K} \to \mathbf{C}) \cong \varprojlim_{\mathbf{J}} \varinjlim_{\mathbf{K}} \hom_{\mathbf{C}}(D(j), E(k)). \qquad (*)$$

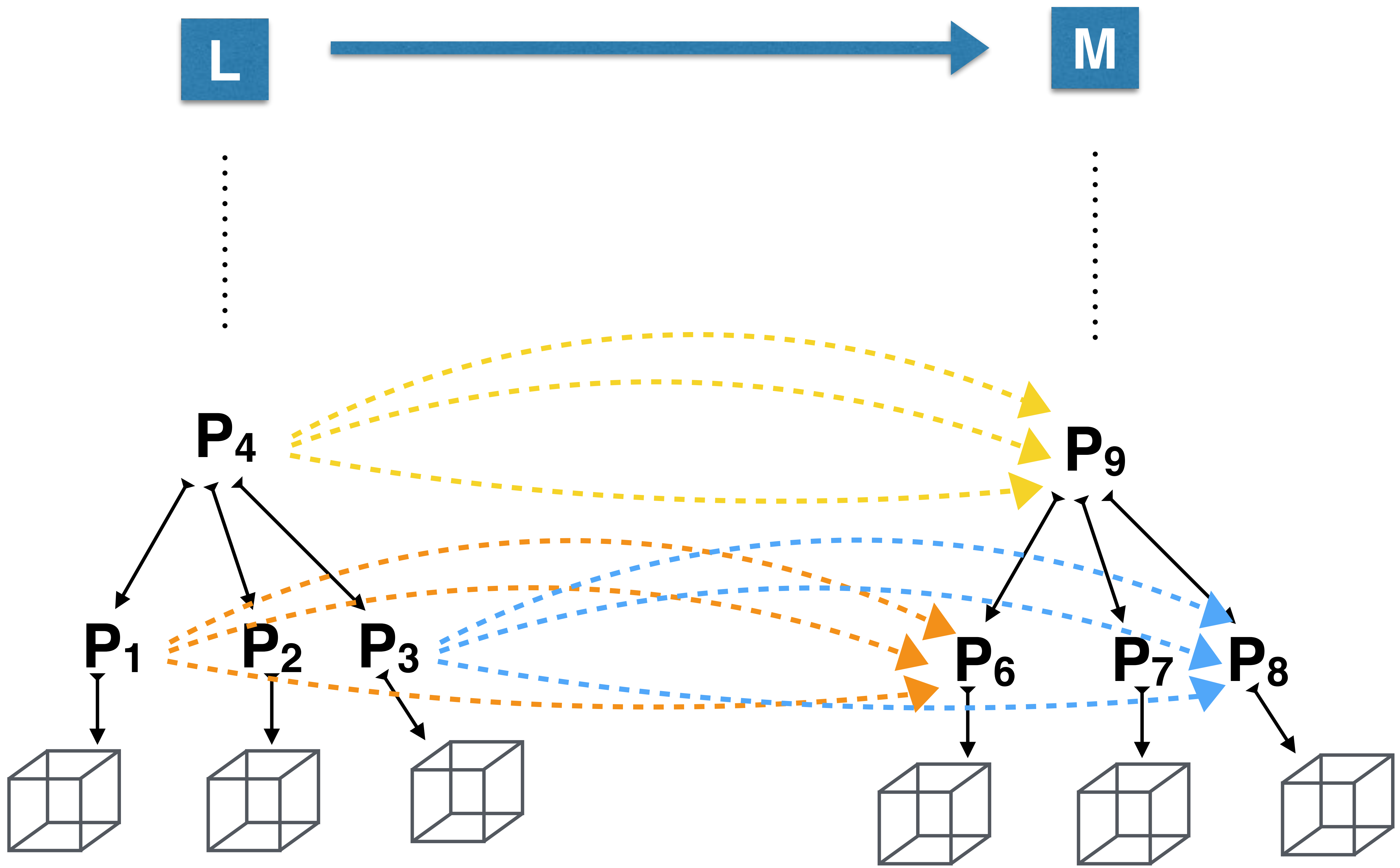
Explicitly, a morphism $f : D \to E$ of ind-objects is a family $(f_j \mid j \in \text{ob}\,\mathbf{J})$, where each f_j is an equivalence class of morphisms from $D(j)$ to objects in the image of E (two such morphisms $D(j) \to E(k)$ and $D(j) \to E(k')$ being equivalent iff there exist morphisms $k \to k''$ and $k' \to k''$ in $\mathbf{K}$ such that



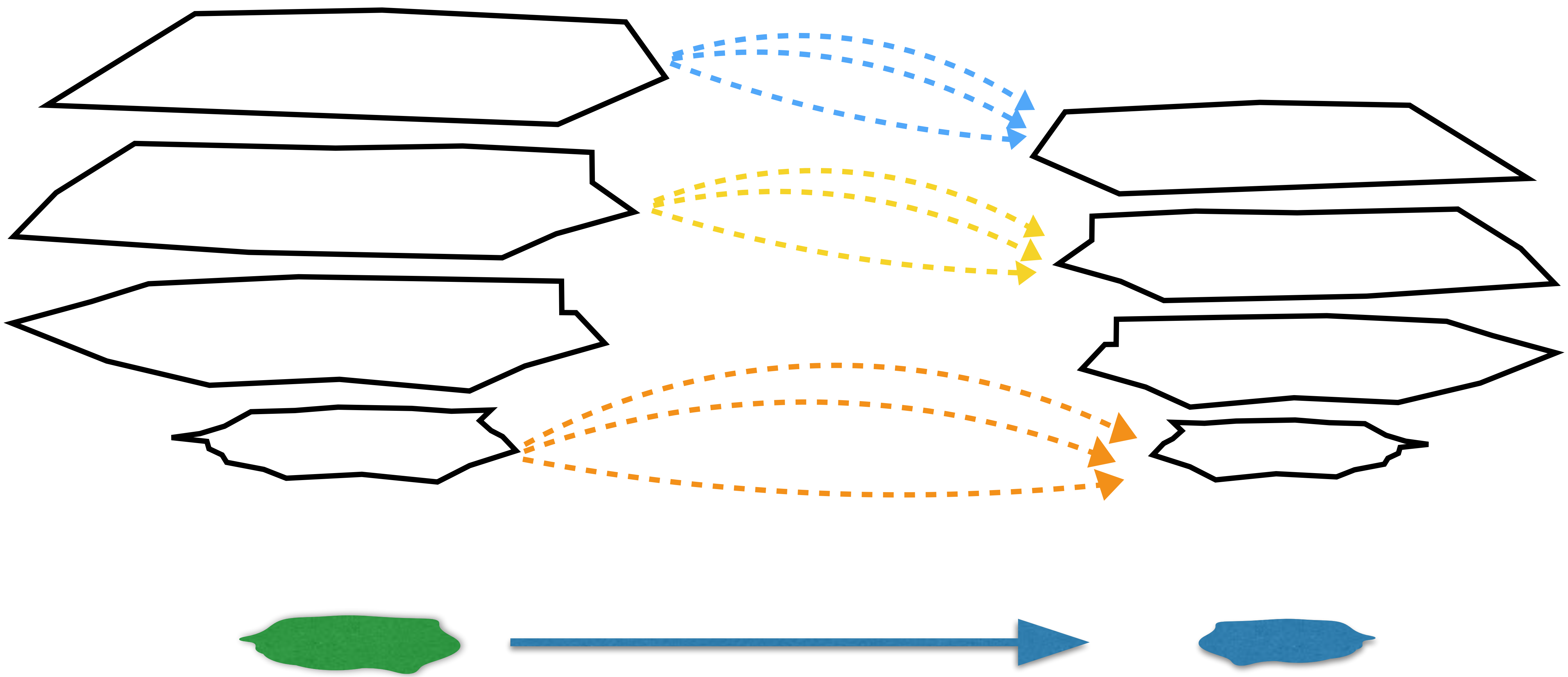
commutes), satisfying the compatibility condition that if $j \to j'$ is a morphism of $\mathbf{J}$ and $D(j') \to E(k)$ belongs to $f_{j'}$, then the composite $D(j) \to D(j') \to E(k)$ belongs to f_j. We leave it to the reader to work out the appropriate definition of composition for these morphisms.

Fortunately, we shall not have to use this explicit description of morphisms of ind-objects very often; but the 'double-limit' description (*) of its hom-sets will be useful in elucidating many of the properties of the category Ind-$\mathbf{C}$ of ind-objects of $\mathbf{C}$. From the process by which we arrived at (*), we have an immediate

Arrows in the pro-completion



Arrows in the finitely generated case

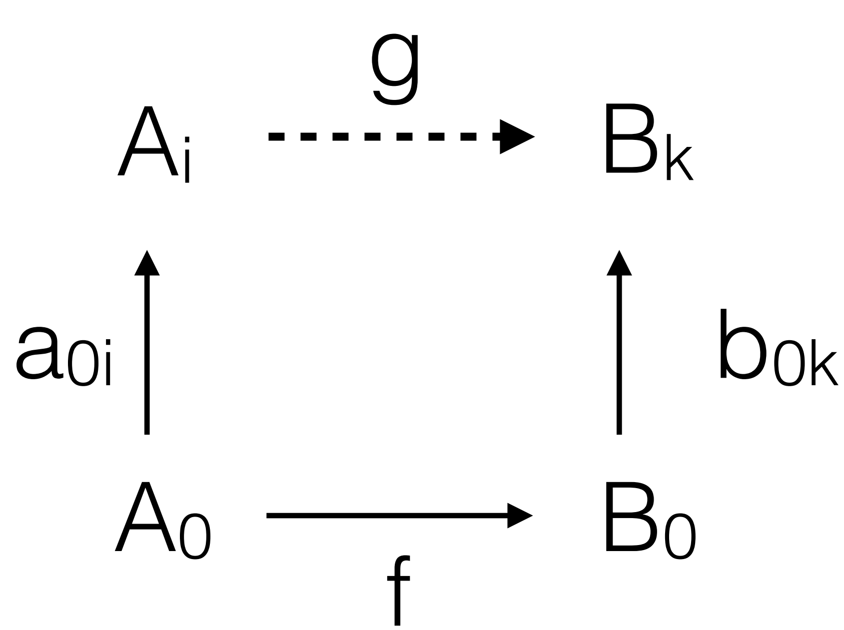


Compatible arrows

$\{(A_i, a_{ij}) \mid i,j \in \omega\}$ Diagrams of f.p. algebras

$\{(B_k, b_{kl}) \mid k,l \in \omega\}$ $A_0=[0,1]^n$, $B_0=[0,1]^m$.

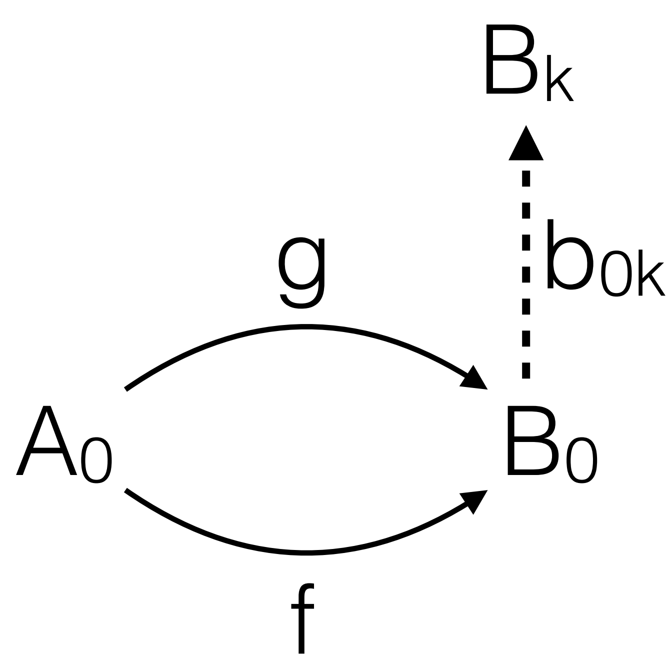
The family of **compatible arrows** C(A,B) is given by all arrows $f : A_0 \longrightarrow B_0$ such that:



Eventually equal maps

Define an equivalence relation E (to be read as f and g being **eventually equal**) on C(A,B) as follows.

Two arrows f, g ∈ C(A,B) are in E, if



The case of finitely generated algebras

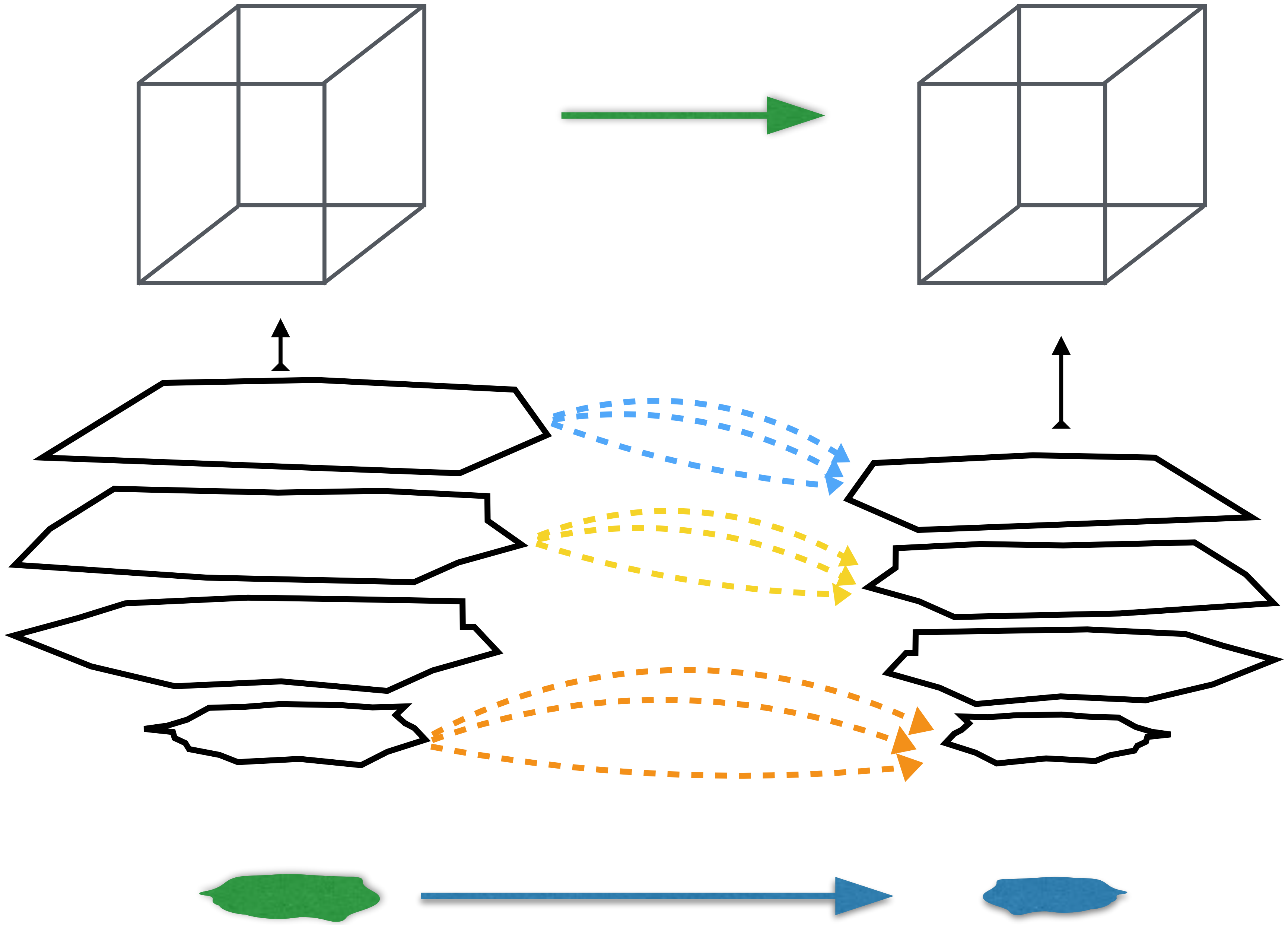
Theorem *Let* $\{(A_i, a_{ij}) \mid i, j \in I\}$ *and* $\{(B_{kl}, b_{kl}) \mid k, l \in K\}$ *be diagrams of order type* ω *in a category* C*,* A *and* B *their respective limits in* $\mathsf{ind\text{-}C}$*, and suppose that the arrows* a_{ij} *and* b_{kl} *are epic.*

1. *For any* $\mathcal{E}$*-equivalence class* C *in* $\mathcal{C}\,(A, B)$ *of arrows* $f\colon A_0 \to B_0$ *there is a corresponding arrow* ϕ_C *between* A *and* B *in* $\mathsf{ind\text{-}C}$*.*

2. *Vice-versa, for any arrow* $\phi = \{\phi_i\}_{i \in I}$ *in* $\mathsf{ind\text{-}C}$ *between* A *and* B*, there is an* $\mathcal{E}$*-equivalence class* C_ϕ *of arrows* $f\colon A_0 \to B_0$ *in* $\mathcal{C}\,(A, B)$*.*

3. *The above associations are such that* $C = C_{\phi_C}$ *and* $\phi = \phi_{C_\phi}$*.*

Arrows in the finitely generated case



The case of finitely generated algebras

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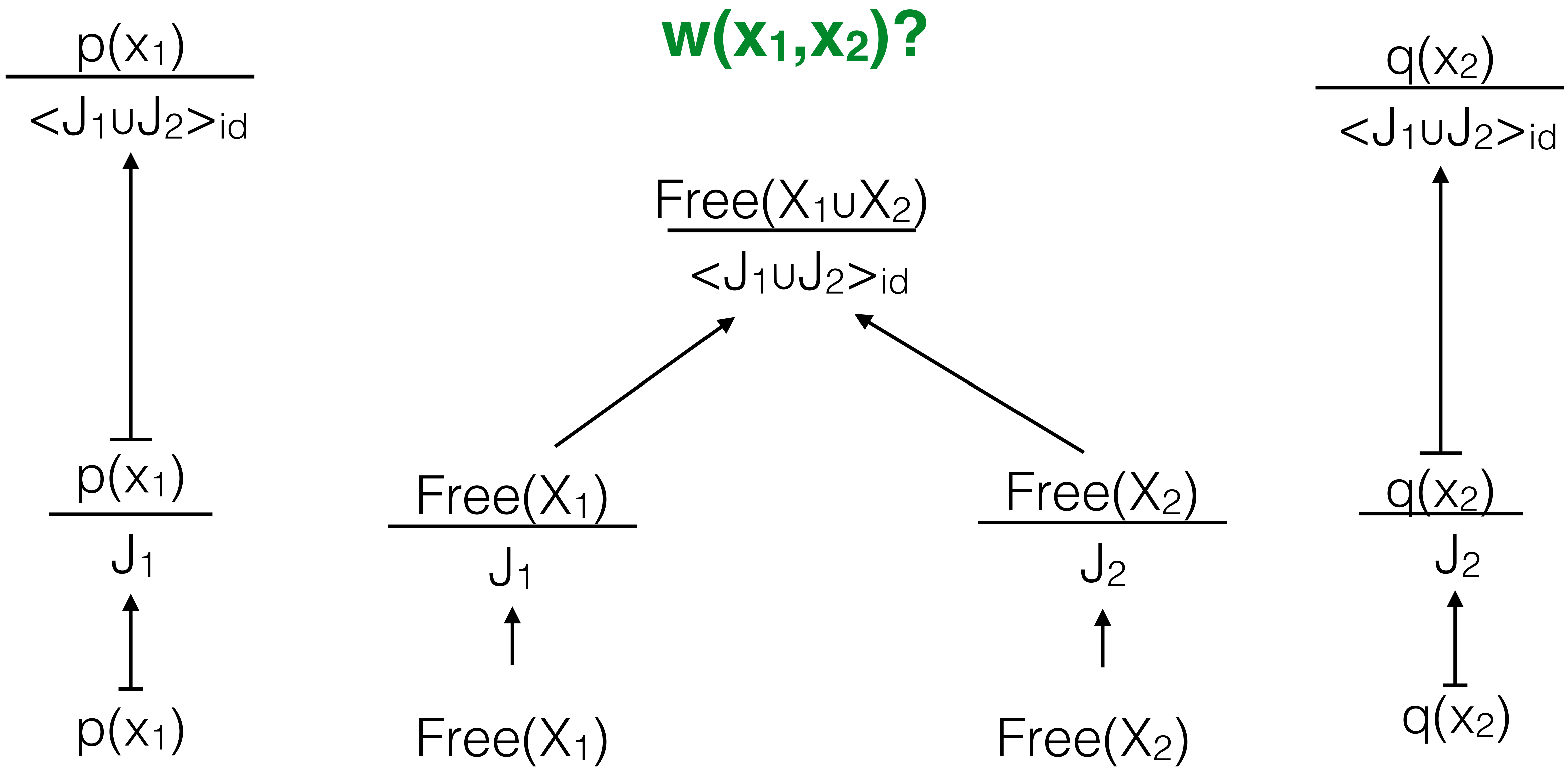
2. *Vice-versa, for any arrow* $\phi = \{\phi_i\}_{i \in I}$ *in* ind-C *between* A *and* B, *there is an* $\mathcal{E}$*-equivalence class* C_ϕ *of arrows* $f\colon A_0 \to B_0$ *in* $\mathcal{C}(A, B)$.

3. *The above associations are such that* $C = C_{\phi_C}$ *and* $\phi = \phi_{C_\phi}$.

How epic are arrows in the general diagram?



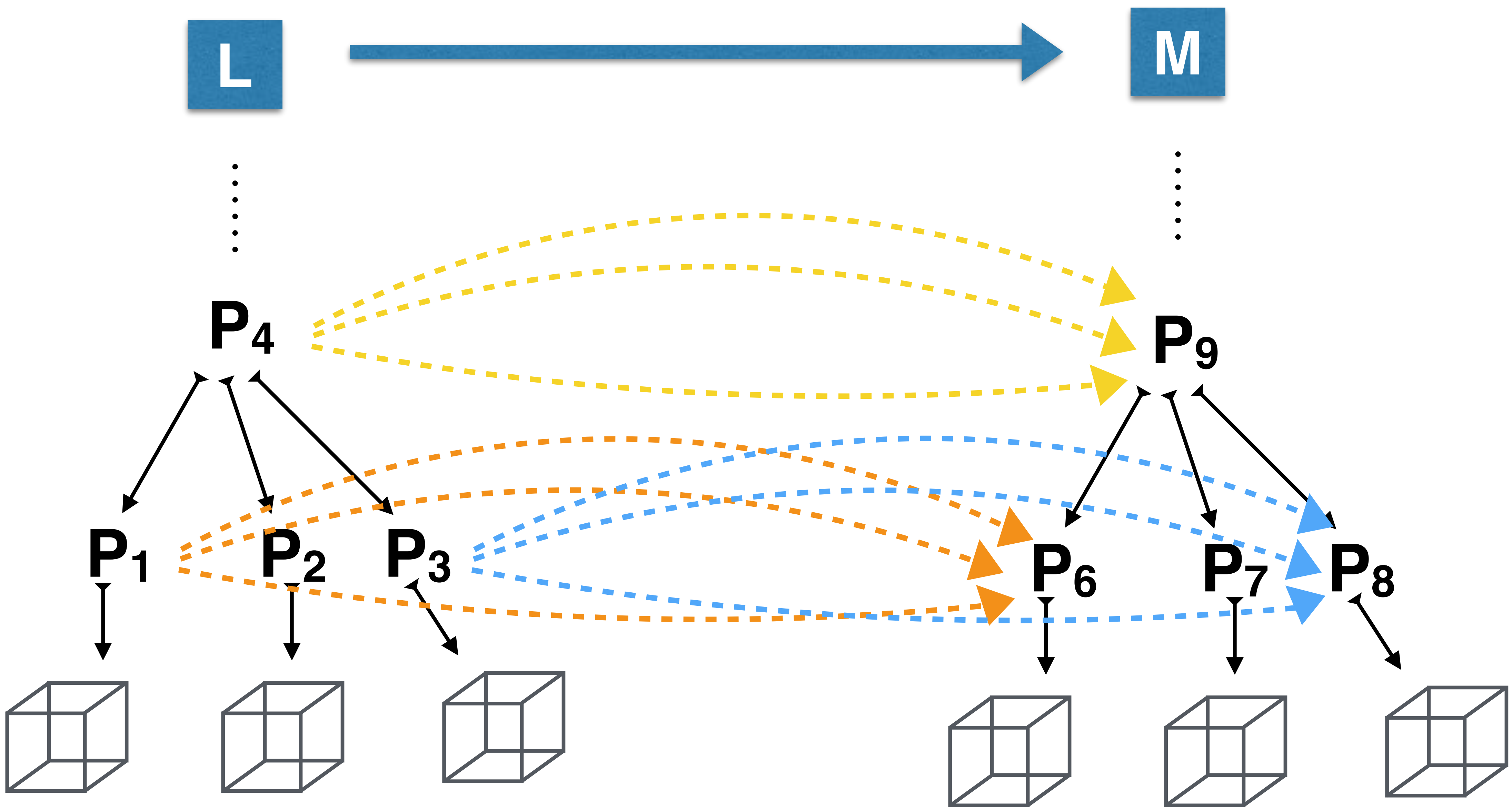
Arrows are ***jointly*** epic

The case of finitely generated algebras

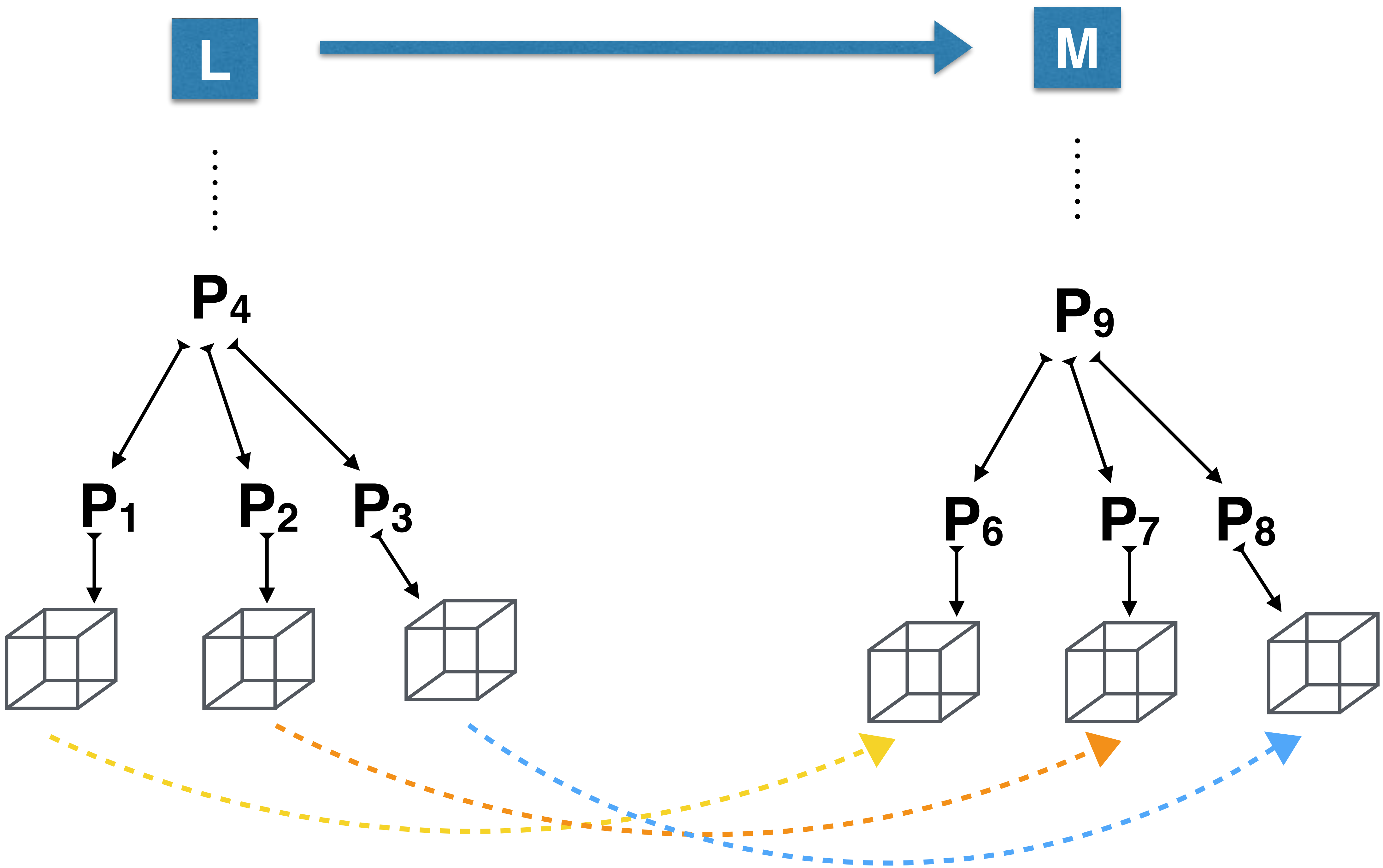
Let us call **inceptive** the objects in a diagram who are not the codomain of any arrow in the diagram

Lemma.
Let C be a finitary algebraic category. Every directed diagram in C_{fp} is isomorphic to a diagram where the **inceptive objects are free algebras** and **transition maps are jointly epic**.

Arrows in the pro-completion



Arrows in the pro-completion



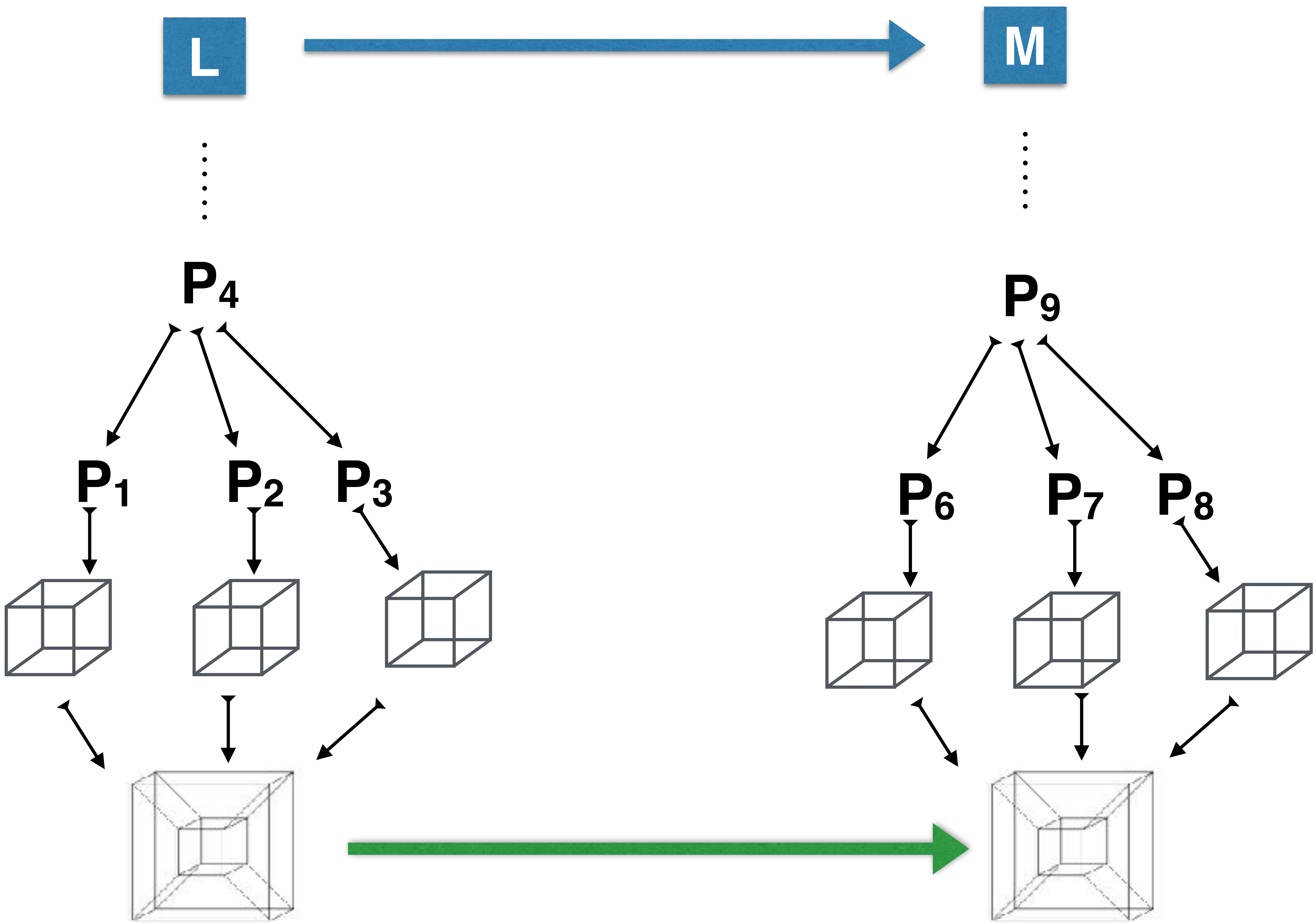
A duality for all MV-algebras

Theorem:
The category of MV-algebras

is dually equivalent to

the category whose objects are **directed diagrams of rational polyhedra** and arrows are $\mathbb{Z}$**-maps** between their inceptive objects.

Arrows in the pro-completion



Open problems

- Can these approximating diagrams be given a more concrete description? (Ongoing research with Sara Lapenta on piecewise geometry on ultrapowers of $\mathbb{R}$.)

- Can the embedding into Tychonoff cubes be made more intrinsic? (Recent joint research with Vincenzo Marra on axioms for *arithmetic separation*.)

- Characterise the topological spaces that arise as the spectrum of prime ideals of MV-algebras. (See the recent preprint by Fred Wehrung solving the problem for second countable spaces.)

- Is it decidable whether two arbitrary finitely presented MV-algebras are isomorphic? (See the work of Daniele Mundici in the last years aiming at attaching computable invariants to rational polyhedra.)