Lukasiewicz logic,
with coefficients

Luca Spada

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Piecewise
linear
geometry

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results

Lukasiewicz logic, with coefficients

Based on a joint work with A. Di Nola, G. Lenzi and V. Marra.

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Łukasiewicz logic

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Łukasiewicz infinite-valued propositional logic is a system in which:

1. truth values are allowed to range in the real interval [0,1],

2. connectives are $\vee$, $\wedge$, $*$, $\rightarrow$, $\neg$.

3. all connectives are continuous (w.r.t. the Euclidean topology on [0,1]).

The idea in a nutshell

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- The main aim is to enhance the expressive power of Łukasiewicz logic by adding truth modifiers to the language.
- They can be understood as linguistic modifiers such as very, particularly, really, etc.
- E.g.

  This tower is high $\mapsto$ This tower is very high.

Rational polyhedra

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Remark

Łukasiewicz logic is the logic of rational polyhedra, in the sense that there is an (effective) bijective correspondence between finite theories in Łukasiewicz logic and rational polyhedra.

A rational polytope is the convex hull of a finite set of rational points in some Euclidean space $\mathbb{R}^n$. Rational polyhedra are finite unions of rational polytopes.

MV-algebras

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Remark

MV-algebras are the equivalent algebraic semantics of Łukasiewicz logic.

An MV-algebra is a structure $\langle A, \oplus, \neg, 0 \rangle$ such that

1. $\langle A, \oplus, 0 \rangle$ is a commutative monoid,
2. $\neg\neg x = x$,
3. $\neg 0 \oplus x = \neg 0$
4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Mundici's functor

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An abelian ℓ-group with order unit ($u\ell$-group, for short), is an ordered Abelian group G whose order is a lattice and that possesses an element u such that

for all $g \in G$, there exists $n \in \mathbb{N}$ such that $(n)u \geq g$.

The functor Γ that takes an $u\ell$-group $\langle G, u \rangle$ to its **unital interval** $[0, u]$ with operation $\oplus$ and $\neg$ defined as follows:

$$x \oplus y = \min\{u, x + y\} \qquad \text{and} \qquad \neg x = u - x,$$

is full, faithful, and dense; hence

Theorem (Mundici 1986)

The category of MV-algebras with their morphisms is equivalent to the category of $u\ell$-groups with ordered group morphisms preserving the order unit.

Piecewise linear geometry with coefficients in $\mathbb{Z}$

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Definition

A $\mathbb{Z}$-map is a continuous and piecewise (affine) linear map from $\mathbb{R}^I$ into $\mathbb{R}^J$, for I, J arbitrary sets, such that each (affine) linear piece has integer coefficients.

Theorem (Marra, – 2012)

The category of finitely presented MV-algebras and the category of rational polyhedra with $\mathbb{Z}$-maps among them are dually equivalent.

So, MV-algebras provide an equational theory to deal with polyhedral geometry, as long as we consider maps with integer coefficients.

Piecewise linear geometry with coefficients in $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$

Rings	alg. closed field k	Algebraic varieties	Polynomial maps
Boolean algebras	$\{0,1\}$	Stone spaces	Continuous maps
Riesz spaces	$\mathbb{R}$	Polyhedra	Piecewise linear maps
MV-algebras	$\mathbb{Z}$	Rational polyhedra	$\mathbb{Z}$-maps
??MVC-algebras	$\mathbb{Z} \subseteq R \subseteq \mathbb{R}$	??R^q-Polyhedra	??R-maps

MVC-algebras

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Duality

Let, for the rest of the talk, R be a fixed but arbitrary ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$ and let $C := \Gamma(R, 1)$.

Definition

An **MVC-algebra** is an MV-algebra $\langle A, \oplus, \neg, 0 \rangle$ endowed with a family of unary operations $\{ f_c \mid c \in C \}$ satisfying the following axioms for all $a, b \in C$ and for all $x, y \in A$.

1. $f_a(x \ominus y) = f_a(x) \ominus f_a(y)$.
2. $f_{a \ominus b}(x) = f_a(x) \ominus f_b(x)$.
3. $f_a(f_b(x)) = f_{a \cdot b}(x)$.
4. $f_1(x) = x$.

Where $\ominus$ is a derived operation defined as $x \ominus y := \neg(\neg x \oplus y)$

Extending the functor Γ

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Theorem

*The functor Γ from uℓ-groups to MV-algebras **can be extended** to a functor from uℓ-modules over R to MVC-algebras. The extended Γ functor form an **equivalence of categories**.*

Completeness

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Theorem

Let R^q be the field of quotients of R and let $C^q = \Gamma(R^q, 1)$. The variety of MVC-algebras is generated by C^q.

Idea of the proof

- If $A \not\models s \approx t$ for an MVC-algebra A, then also $M \not\models s \approx t$ with $\Gamma(M, u) = A$. W.l.o.g. take A l.o.

- The uℓ-module M embeds in a torsion free, R-divisible, totally ordered module.

- The first order theory of these R-modules enjoys EQ and it is complete, hence all its models are elementarily equivalent.

- The failure of an equation in M can be encoded in a first order formula, hence the equation also fails in the uℓ-module R^q. So it fails in C^q.

McNaughton theorem for MVC-algebras

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McNaughton showed that the terms definable in n in the language of MV-algebras are exactly the $\mathbb{Z}$-maps from $[0,1]^n$ into $[0,1]$.

Theorem

*The free k-generated MVC-algebra is isomorphic to the algebra of **piecewise linear functions** from $[0,1]^k$ into $[0,1]$ such that each affine linear piece has **coefficients in R**.*

Duality

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Theorem

There is a categorical duality between

MVC-algebras,		Polyhedra with vertices in R^q,
	and	
MVC-homomorphisms		Piecewise linear functions with coefficients in R.

Thank you for your attention!