Dualities for MV-algebras

In memoriam Leo Esakia.

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Aim of the talk

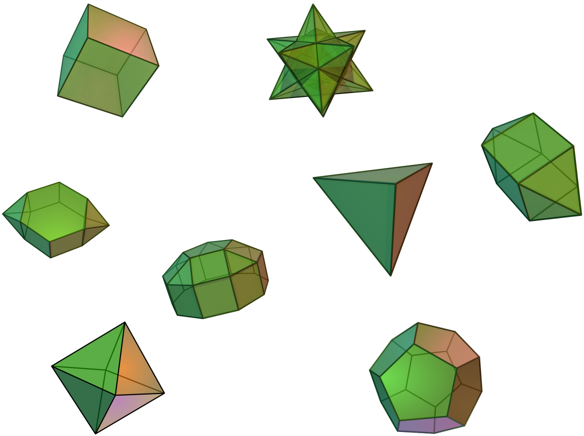
Either it is soft or not.

If an apple is red or yellow, then put it in the first basket.

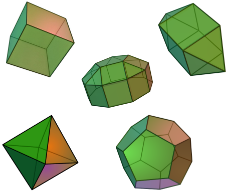
The higher a men is, the easier is that he is blond

If is too cold increase heating a bit.

Aim of the talk



Aim of the talk



Categorical duality

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Space and numbers

René Descartes
1596 – 1650

There exists a correspondence between points in the space and tuples of numbers.

An formal correspondence

David Hilbert
1862 – 1943

The correspondence can be lifted to subspaces and set of equations.

$$\mathbb{V}\Big(p(\bar{x}) = q(\bar{x})\Big) = \big\{(a_1, .., a_n) \mid p(a_1, .., a_n) = p(a_1, .., a_n)\big\}$$

and

$$\mathbb{I}\Big((a_1, .., a_n)\Big) = \big\{p(\bar{x}) = q(\bar{x}) \mid p(a_1, .., a_n) = q(a_1, .., a_n)\big\}.$$

Totally disconnected spaces

Marshall Stone
1903 – 1989

Every boolean algebra is isomorphic to the algebra of clopen sets of a totally disconnected, compact, Hausdorff space.

$$A \mapsto \mathrm{Spec}(A) = \{M \subseteq A \mid M \text{ is an maximal filter}\}.$$

Totally disconnected spaces

Marshall Stone
1903 – 1989

Every boolean algebra is isomorphic to the algebra of clopen sets of a totally disconnected, compact, Hausdorff space.

$$\frac{\mathcal{F}_\mu}{I} \cong A \mapsto \{\mathbf{a} \in \{0,1\}^\mu \mid t(\mathbf{a}) \approx 1 \text{ for any } t \in I\}.$$

MV-algebras

Łukasiewicz logic

Jan Łukasiewicz
1878 – 1956

Łukasiewicz logic is a many-valued logical system involving the basic connectives $\to$ (implication) and $\neg$ (negation), and axiomatised by the four axiom schemata:

1. $\alpha \to (\beta \to \alpha)$,
2. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$,
3. $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$,
4. $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$,

with *modus ponens* as the only deduction rule.

Semantics of Łukasiewicz logic

Łukasiewicz logic is a subsystem of classical logic and has a many-valued semantics: assignments μ to atomic formulæ range in the unit interval $[0,1] \subseteq \mathbb{R}$.
They are extended compositionally to compound formulæ via

$$\begin{gathered} \mu(\neg\alpha) \;=\; 1 - \mu(\alpha)\,, \\ \mu(\alpha \to \beta) = \min\{1, 1 - \mu(\alpha) + \mu(\beta)\} \end{gathered}$$

Tautologies are defined as those formulæ that evaluate to 1 under every such assignment.

MV-algebra

Chen Chung Chang
19?? –

In 1958 Chang considered the Tarski-Lindenbaum algebras of Łukasiewicz logic and called them MV-algebras.

Definition

An MV-algebra is a structure $\mathcal{A} = \langle A, \oplus, ^*, 0\rangle$ such that:

- $\mathcal{A} = \langle A, \oplus, 0\rangle$ is a comm. monoid,
- * is an involution, i.e. $(x^*)^* = x$,
- the rules of interaction between $\oplus$ and * are given by:
  - $x \oplus 0^* = 0^*$
  - $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$

Example 1: The standard MV-algebra

In modern terms one says that MV-algebras are the equivalent algebraic semantics of Łukasiewicz logic.

Example

Consider the set of real number $[0,1]$ endowed with the following operation:

$$\neg x = 1 - x \text{ and } x \oplus y = \min\{1, x + y\} \text{ (truncated sum).}$$

Then $\langle [0,1], \oplus, \neg, 0 \rangle$ is an MV-algebra.

Actually the above algebra generates the variety of all MV-algebras. So the equations that hold for any MV-algebra are exactly the ones that hold in [0,1].

Example 2: lattice ordered groups

A (commutative) lattice ordered group G, ℓ-group for short, is a (commutative) group $\langle G, +, -, 0\rangle$ with an order $\leqslant$ which is compatible with the operation $+$, i.e.

$$x \leqslant y \qquad \text{implies} \qquad x + z \leqslant y + z\,.$$

If g is a positive element in G, then the interval $[0, g]$, endowed with the truncation of the operations is an MV-algebra. So, for instance, the MV-algebra above is obtained from the ℓ-group $\langle \mathbb{R}, +, -, 0\rangle$ by taking $g = 1$. Actually, every MV-algebra can be obtained as the interval of some ℓ-group.

Definable function

The fact that $[0,1]$ has a standard structure of MV-algebra will be tacitly used throughout the presentation.

Definition

Given $S \subseteq [0,1]^{\mu}$ and $T \subseteq [0,1]^{\nu}$, a function $\lambda \colon S \to T$ is definable if there exists a ν-tuple of terms $(l_\beta)_{\beta<\nu}$, with $l_\beta \in \mathcal{F}_\mu$, such that

$$\lambda(\mathbf{p}) = (\, l_\beta(\mathbf{p})\,)_{\beta<\nu}$$

for every $\mathbf{p} \in S$. Any such ν-tuple is called a family of defining terms for λ.

The functors $\mathscr{I}$ and $\mathscr{V}$

The basic adjunction

I shall consider the category $\mathsf{T}_{\mathsf{def}\,\mathbb{Z}}$ of subspaces of the Tychonoff cubes $[0,1]^\mu$, with definable maps as morphisms.

Further, let $\mathsf{MV_p}$ be the category of presented MV-algebras, i.e. MV-algebra of the form $\frac{\mathcal{F}_\mu}{\theta}$, where $\mathcal{F}_\mu$ is the free MV-algebra on some cardinal μ and θ is a congruence of $\mathcal{F}_\mu$, together with their homomorphisms.

The basic adjunction

My aim for the moment is to construct a pair of functors

$$\mathscr{I}: \ \mathsf{T}^{op}_{\mathrm{def}\,\mathbb{Z}} \longrightarrow \mathsf{MV_p} \ , \quad \mathscr{V}: \ \mathsf{MV_p} \longrightarrow \mathsf{T}^{op}_{\mathrm{def}\,\mathbb{Z}} \ .$$

The functor $\mathscr{I}$: Objects.

Given $S \subseteq [0,1]^\mu$, let, for arbitrary terms $s, t \in \mathcal{F}_\mu$,

$$(s,t) \in \mathbb{I}\,(S) \text{ if and only if } [0,1] \models s(\mathbf{p}) \approx t(\mathbf{p})$$

for every $\mathbf{p} \in S \subseteq [0,1]^\mu$.

Remark

The defining condition for $\mathbb{I}\,(S)$ is equivalent to for any $\forall \mathbf{p} \in S \quad s(\mathbf{p}) = t(\mathbf{p})$ as real numbers.

As for any $S \subseteq [0,1]^\mu$, it is easy to check that $\mathbb{I}\,(S)$ is a congruence on $\mathcal{F}_\mu$ one may define

$$\boxed{\mathscr{I}\,(S) \;=\; \mathcal{F}_\mu \,/\, \mathbb{I}\,(S)\;.}$$

The functor $\mathscr{I}$: Arrows.

Given $S \subseteq [0,1]^\mu$ and $T \subseteq [0,1]^\nu$, let $\lambda\colon S \to T$ be a definable map, and let d be a ν-tuple of defining terms for λ. Then there is an induced function

$$\mathscr{I}(\lambda)\colon \mathscr{I}(T) \to \mathscr{I}(S)$$

which acts on each $s \in \mathcal{F}_\nu$ by substitution as follows:

$$\boxed{\frac{s(x)}{\mathbb{I}(T)} \in \mathscr{I}(T) \quad \stackrel{\mathscr{I}(\lambda)}{\longmapsto} \quad \frac{s(x/d)}{\mathbb{I}(S)} \in \mathscr{I}(S)\ .}$$

Remark on well-definition

Remark

1. There can be several distinct defining terms for a definable function $\lambda\colon S \to [0,1]$. However, d and d' are defining terms for the same function λ if and only if $(d, d') \in \mathbb{I}(S)$.

2. Further, the definition of $\mathscr{I}(\lambda)$ above does not depend on the choice of the representing term s, for if s' is another term such that $(s, s') \in \mathbb{I}(T)$, then $s\big([X_\beta \backslash l_\beta]_{\beta<\nu}\big)$ is congruent to $s'\big([X_\beta \backslash l_\beta]_{\beta<\nu}\big)$ modulo $\mathbb{I}(S)$, because substitutions commute with congruences.

The functor $\mathscr{V}$: Objects.

Given $R = \{(s_i, t_i) \mid i \in I\} \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$, for I an index set, the vanishing locus of R is

$$\mathbb{V}(R) = \{p \in [0,1]^\mu \mid [0,1] \models s_i(p) \approx t_i(p) \text{ for each } i \in I\}.$$

As for any congruence θ on $\mathcal{F}_\mu$ we have $\mathbb{V}(\theta) \subseteq [0,1]^\mu$ we set

$$\boxed{\mathscr{V}(\mathcal{F}_\mu/\theta) = \mathbb{V}(\theta).}$$

The functor $\mathscr{V}$: Arrows.

For each $\alpha < \mu$, let π_α be the α^{th} projection term.

Let $h\colon \mathcal{F}_\mu/\theta_1 \to \mathcal{F}_\nu/\theta_2$ be a homomorphism of MV-algebras.
Fix, for each α, an arbitrary $f_\alpha \in h(\pi_\alpha/\theta_1)$.

For any $\mathbf{p} \in \mathbb{V}(\theta_2)$, set

$$\mathscr{V}(h)(\mathbf{p}) \;=\; \big(\, f_\alpha(\mathbf{p}) \,\big)_{\alpha<\mu}\,.$$

The basic adjunction

A Galois adjunction

Lemma

For each $S \subseteq [0,1]^\mu$ and $R \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$,

$$R \subseteq \mathbb{I}(S) \text{ if, and only if, } S \subseteq \mathbb{V}(R)\ .$$

In other words, the functions $\mathbb{V}$ and $\mathbb{I}$ form a (isotone) Galois connection.

Consequences of the Galois adjunction

Corollary

Let $S_1, S_2 \subseteq [0,1]^\mu$ and $R_1, R_2 \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$, then

a) *The functions $\mathbb{I}$ and $\mathbb{V}$ are isotone i.e. $S_1 \subseteq S_2$ implies $\mathbb{I}(S_2) \subseteq \mathbb{I}(S_1)$ and $R_1 \subseteq R_2$ implies $\mathbb{V}(R_2) \subseteq \mathbb{V}(R_1)$.*

b) *The compositions $\mathbb{I}\,\mathbb{V}$ and $\mathbb{V}\,\mathbb{I}$ are extensive, i.e. $S_1 \subseteq \mathbb{V}(\mathbb{I}(S_1))$ and $R_1 \subseteq \mathbb{I}(\mathbb{V}(R_1))$.*

c) *The compositions $\mathbb{I}\,\mathbb{V}$ and $\mathbb{V}\,\mathbb{I}$ give fixed points, i.e. $\mathbb{I}(\mathbb{V}(\mathbb{I}(S_1))) = \mathbb{I}(S_1)$ and $\mathbb{V}(\mathbb{I}(\mathbb{V}(R_1))) = \mathbb{V}(R_1)$.*

The unit

A unit $\eta\colon \mathbb{1}_{\mathsf{MV_p}} \to \mathscr{I}\mathscr{V}$ exists

$$
\begin{tikzcd}
\mathscr{I}\mathscr{V}\left(\frac{\mathcal{F}_\mu}{\theta_1}\right) \arrow[r, "\mathscr{I}\mathscr{V}(h)"] & \mathscr{I}\mathscr{V}\left(\frac{\mathcal{F}_\nu}{\theta_2}\right) \\
\frac{\mathcal{F}_\mu}{\theta_1} \arrow[u, "\eta_{\frac{\mathcal{F}_\mu}{\theta_1}}"] \arrow[r, "h"] & \frac{\mathcal{F}_\nu}{\theta_2} \arrow[u, "\eta_{\frac{\mathcal{F}_\nu}{\theta_2}}"']
\end{tikzcd}
$$

The co-unit

A a co-unit $\varepsilon\colon \mathscr{V}\,\mathscr{I} \to \mathbb{1}_{\mathsf{T}^{op}_{\mathrm{def}\,\mathbb{Z}}}$ exists.

$$
\begin{tikzcd}
\mathscr{V}\,\mathscr{I}\,(S) \arrow[r, "\mathscr{V}\,\mathscr{I}\,(\lambda)"] \arrow[d, "\varepsilon_S"'] & \mathscr{V}\,\mathscr{I}\,(T) \arrow[d, "\varepsilon_T"] \\
S \arrow[r, "\lambda"] & T
\end{tikzcd}
$$

Naturality

Further, for any $A \in \mathsf{MV_p}$:

$$\begin{tikzcd} \mathscr{V}(A) \arrow[r, "\mathscr{V}(\eta_A)"] \arrow[rr, bend right, "\mathbb{1}_{\mathscr{V}(A)}"'] & \mathscr{V}\mathscr{I}\mathscr{V}(A) \arrow[r, "\varepsilon_{\mathscr{V}(A)}"] & \mathscr{V}(A) \end{tikzcd},$$

and for any $K \in \mathsf{T}^{op}_{\mathrm{def}\,\mathbb{Z}}$:

$$\begin{tikzcd} \mathscr{I}(K) \arrow[r, "\mathscr{I}(\varepsilon_K)"] \arrow[rr, bend right, "\mathbb{1}_{\mathscr{I}(K)}"'] & \mathscr{I}\mathscr{V}\mathscr{I}(K) \arrow[r, "\eta_{\mathscr{I}(K)}"] & \mathscr{I}(K) \end{tikzcd}.$$

The basic adjunction

Theorem

$\mathscr{V} \dashv \mathscr{I}$, in words, the functor $\mathscr{V}$ is left adjoint to the functor $\mathscr{I}$.

It should be noted at this point that subspaces of the cubes are in 1-1 correspondence with compact Hausdorff spaces. However, just as a concrete presentation of the MV-algebra is needed to associate the space, also a concrete embedding of a compact Hausdorff space into the cube is needed for definable functions to make sense.

Semi-simple MV-algebras

Semi-simple MV-algebras

Recall that

Definition

An algebra is called simple, if it has no non-trivial congruences (equivalently, if it is a quotient over a maximal congruence).

Further,

Definition

An algebra is called semi-simple if it is the subdirect product of simple algebras.

The fixed points of the adjunction

Point=Maximal congruence

For any set $S \subseteq [0,1]^{\mu}$, and for any congruence θ on $\mathcal{F}_{\mu}$, the following hold.

1. If θ is a maximal congruence then $\mathbb{V}(\theta)$ is a singleton.
2. If S is a singleton then $\mathbb{I}(S)$ is a maximal congruence.

Proposition (Nullstellensatz)

1. For any $R \subseteq \mathcal{F}_{\mu} \times \mathcal{F}_{\mu}$, $\mathbb{I}(\mathbb{V}(R)) = \mathrm{Rad}(\mathcal{F}_{\mu}/\langle R \rangle) = \bigcap\{\theta \text{ max. cong. in } \mathcal{F}_{\mu} \mid R \subseteq \theta\}$
2. The algebra $\mathcal{F}_{\mu}/\theta$ is semi-simple if, and only if, $\mathbb{I}(\mathbb{V}(\theta)) = \theta$.

The fixed points of the adjunction

Complete regularity by definable functions

For any point $p \in [0,1]^\mu$ and any closed set $K \subseteq [0,1]^\mu$ with $p \notin K$, there is a definable function $\lambda\colon [0,1]^\mu \to [0,1]$ that takes value 0 over K, and value > 0 at p.

Co-Nullstellensatz

1. For any $S \subseteq [0,1]^\mu$, $\mathbb{V}(\mathbb{I}(S)) = \mathsf{cl}(S)$.
2. The set $S \subseteq [0,1]^\mu$ is closed if, and only if, $\mathbb{V}(\mathbb{I}(S)) = S$.

A duality for semi-simple MV-algebras

The above propositions immediately give:

Theorem

The category MV^{ss}_{p} is dually equivalent to the full subcategory of $\mathsf{T}_{\mathrm{def}\,\mathbb{Z}}$ given by closed subspaces of the cubes.

Finitely presented MV-algebras

Finitely presented MV-algebras

Definition

A (presented) MV-algebra $\mathcal{F}_\mu/\theta$ is finitely presented if μ is a non-negative integer, and θ is a finitely generated congruence, equivalently, θ is a compact element in the algebraic lattice of congruences on A.

Lemma

Let s, t, u, v be elements of $\mathcal{F}_m$, then $(u, v) \in \langle (s, t) \rangle$ if, and only if, $\mathbb{V}(s, t) \subseteq \mathbb{V}(u, v)$.

The proof of this lemma involves a geometric argument, Chang's Completeness Theorem, and the easily proved fact that definable functions are piecewise linear maps.

Wójcicki's Theorem

Every finitely presented MV-algebra is semi-simple.

Finitely definable sets

Definition

A subset $S \subseteq [0,1]^\mu$ is called finitely definable if there is a finite index set I, along with a set of pairs $R = \{(s_i, t_i) \in \mathcal{F}_\mu \times \mathcal{F}_\mu \mid i \in I\}$, such that $S = \mathbb{V}(R)$.

The full subcategory of $\mathsf{T}_{\mathrm{def}\,\mathbb{Z}}$ whose objects are finitely definable subsets of $[0,1]^m$, as m ranges over all non-negative integers, is denoted $\mathsf{D}_{\mathrm{def}\,\mathbb{Z}}$.

Finitely definable set=Compact congruence

1. If D is a finitely definable set, then $\mathbb{I}(D)$ is a finitely generated congruence.
2. If θ is a finitely generated congruence, then $\mathbb{V}(\theta)$ is a finitely definable set.

A duality for finitely presented MV-algebra

So one immediately gets

Theorem

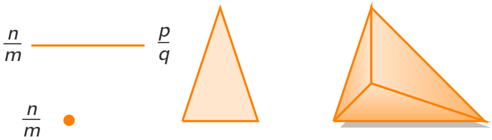
The adjunction $\mathscr{V} \dashv \mathscr{I}$ restricts to an equivalence of categories between $\mathsf{MV}_{\mathsf{fp}}$ *and* $\mathsf{D}^{op}_{\mathsf{def}\,\mathbb{Z}}$.

Rational polyhedral geometry

Finally, the category $\mathsf{D}_{\mathrm{def}\,\mathbb{Z}}$ can be described in a concrete way.

Definition

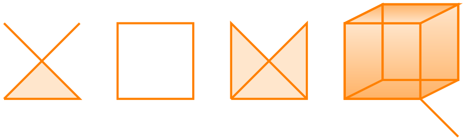
A rational polytope is the convex hull of a finite set of **rational** points.



Rational polyhedral geometry [Cont.d]

Definition

A rational polyhedron is the union of a finite number of rational polytopes.



Definition

A $\mathbb{Z}$-map is a continuous piecewise linear function with integer coefficients.

McNaughton theorem

Theorem (McNaughton 1951)

Let $P \subseteq [0,1]^m$ be a rational polyhedron, and let $\lambda\colon P \to [0,1]$ be any function. Then λ is a $\mathbb{Z}$-map if, and only if, λ is a definable function.

Proposition (folklore)

A set $S \subseteq [0,1]^m$ is a rational polyhedron if, and only if, there is a $\mathbb{Z}$-map vanishing precisely on S.

Duality for finitely presented MV-algebras

Corollary

The category $\mathsf{D}_{\mathrm{def}\,\mathbb{Z}}$ *is the category of rational polyhedra and* $\mathbb{Z}$*-maps.*

Future works

- Can we provide a method to construct such an adjunction for any given variety?
- What happens if we substitute the MV-algebra [0,1] with some non-semi-simple algebra like $\Gamma(\mathbb{Z}\bar{\times}\ldots\bar{\times}\mathbb{Z},(1,0,0,\ldots,0))$ or $[0,1]^*$?
- Can the construction be generalised to cope with prime ideals, rather than maximal ones?

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