ŁΠ Logic with Fixed Point Operator

Luca Spada

Dipartimento di Matematica

Università degli Studi di Siena

e-mail:spada@unisi.it
http://homelinux.capitano.unisi.it/~spadal

Contents of the talk

1 Preliminaries

History

- **Gödel**, **Lukasiewicz** and **Product Logic** are at the basis of the t-norm based Logics
- Hájek found the common intersection of these three logic: **Basic Logic** ([Haj98])
- Two questions:
  - Which one to use to grasp the expressiveness of Fuzzy Logic?
  - What about the *union* of those Logics?

- Esteva, Godo and Montagna introduced **ŁΠ Logic** ([EG99], [Mon00], [EGM01])
- It is a powerful Logic still having good algebraic properties.

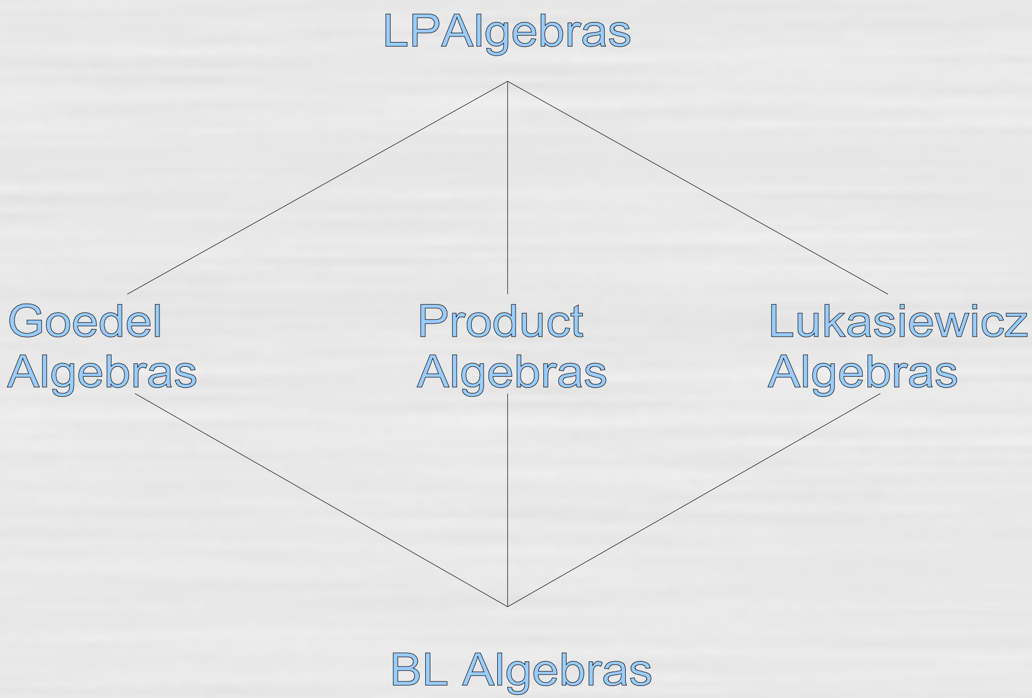

Figure 1: Diagram of principal t-norm based Logics

ŁΠ Algebras

Definition 1.1. An LΠ algebra is a structure

$$\mathcal{L} = \langle L, \oplus, -_L, \Rightarrow_\Pi, *_\Pi, 0_L, 1_L \rangle$$

where :

1. $\langle L, \oplus, -_L, 0_L \rangle$ is a MV algebra
2. $\langle L, \Rightarrow_\Pi, *_\Pi, 0_L, 1_L \rangle$ is a Π algebra
3. $x *_\Pi (y \ominus z) = (x *_\Pi y) \ominus (x *_\Pi z)$
4. $\Delta(x \Rightarrow_L y) \Rightarrow_L (x \Rightarrow_\Pi y) = 1_L$

This definition differs from the original one from [EGM01] and was introduced in [Cin04].
Interesting enough the redundancy of axiom (4) is still open.

Results

Theorem 1.2 (Algebraic Completeness). *ŁΠ $\vdash \varphi$ if, and only if, in any* LΠ *algebra* $\varphi^* = 1$ *holds.*

Theorem 1.3 (Standard Completeness). *ŁΠ $\vdash \varphi$ if, and only if, in the* LΠ *algebra on the real interval* $[0,1]$, $\varphi^* = 1$ *holds.*

Theorem 1.4. *The category of linearly ordered* LΠ *algebras and the category of linearly ordered field are equivalent.*

Theorem 1.5. *The categories of* LΠ *algebras and f-semifield are equivalent.*

Properties of ŁΠ Logic

- Obviously, **Lukasiewicz** and **Product Logic** are faithfully interpretable in ŁΠ
- **Gödel** Logic is faithfully interpretable in ŁΠ
- **Takeuti and Titani's Logic** is faithfully interpretable in ŁΠ$\frac{1}{2}$
- **Pavelka's Rational Logic** is faithfully interpretable in ŁΠ$\frac{1}{2}$
- **Pavelka's Rational Product Logic** is faithfully interpretable in ŁΠ$\frac{1}{2}$

Why stopping here??

2 Fixed Points

History

Fixed Points are presents in many field of Mathematics, Logic and Computer Science

in Logic:

– Extensions of First Order Logic (LFP, PFP, IFP, etc.)
– modal μ calculus

Based on operators defined on First Order structures

Their existence is based on Tarski's fixed point Theorem

In Propositional Logic seems to be no simple solution...

...but in *Many Valued* Propositional Logic one can see formulas as functions! In particular in our case they are *continuous* functions from $[0, 1]$ to itself

Brouwer Theorem

Theorem 2.1 (Brouwer 1910)**.** *Every continuous function from a convex compact to itself has a fixed point*

3 μŁΠ Logic

Some Precautions

- To steek with continuous functions we will not allow fixed points of formulas containing the symbol $\to_\Pi$
- For sake of readability we will use a fresh variable symbol x to denote the variable falling under the scope of the fixed point operator.

μŁΠ Logic

Definition 3.1. The Fixed Point LΠ Logic (μLΠ logic for short) has the following theory:

1. Al the axioms and rules from LΠ Logic
2. $\mu.x(T(x)) \leftrightarrow T(\mu.x(T(x)))$
3. $(\bigwedge_{i \leq n} \Delta(p_i \leftrightarrow q_i)) \to (\mu.x(T(p_1, ..., p_n)) \leftrightarrow \mu.x(T(q_1, ..., q_n)))$
4. The rule $\frac{T(p) \leftrightarrow p}{\mu(T(x)) \to p}$

μŁΠ algebras

Definition 3.2. A μLΠ algebra is a particular LΠ algebra endowed with an operator μ that satisfies the following condition for any term $T(x)$ not containing the symbol $\rightarrow_{\Pi}$.

1. $\mu.x(T(x)) = T(\mu.x(T(x)))$
2. If $T(t) = t$ then $\mu.x(T(x)) \leq t$
3. $(\bigwedge_{i \leq n} \Delta(p_i \Leftrightarrow q_i)) \leq (\mu.x(T(p_1, ..., p_n)) \Leftrightarrow \mu.x(T(q_1, ..., q_n)))$

Proposition 3.3. *μŁΠ algebras form a variety*

Completeness

Theorem 3.4. *μŁΠ Logic is algebraically complete, i.e. $\mu\text{Ł}\Pi \vdash \varphi$ if, and only if, in every μŁΠ algebra holds $\varphi^* = 1$*

Lemma 3.5. *Any μŁΠ algebra is isomorphic to a subdirect product of linearly ordered μŁΠ algebras*

Representation

Theorem 3.6. *Any l.o. μŁΠ algebra is isomorphic to the interval algebra of some real closed field, conversely every l.o. real closed field contains a definable μŁΠ algebra.*

sketch. For any $c \in R$ and any polynomial $p(x)$ such that $p(c) = 0$ we can translate $p(x)$ in a term of the algebra $p^*(x)$. Now, considering the term

$$p'(x) = p^*(x) \oplus x$$

it is sufficient to take $\mu.x(p'(x))$ as the wanted element.
For the other direction just note that the existence of a fixed point for a term of the algebra can represented as a first order formula of the form

$$\forall P \exists x (\bigvee_{i \in I} \bigwedge_{j \in J} P_{ij}(x))$$

Now simply use Quantifier Elimination.... $\square$

Standard Completeness

Theorem 3.7. *$\mu L\Pi$ logic is standard complete, i.e. $\mu Ł\Pi \vdash \varphi$ if, and only if, the $\mu Ł\Pi$ algebra on the real interval $[0,1]$ models $\varphi^* = 0$*

Open questions and Future Works

- μŁΠ Logic is decidable, due to Q.E. in real closed fields, but which is its complexity?
- We allowed one the scope of μ to be just one variable, what happens if we allow more?
- Is possible to extend all the machinery to the first order?

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