Noname manuscript No.
(will be inserted by the editor)

An expansion of Basic Logic with fixed points

Luca Spada

In memoriam Franco Montagna

Abstract We introduce an expansion of Basic Logic (BL) with new connectives which express fixed points of *continuous formulas*, i.e. formulas of BL whose connectives are among $\{\&, \vee, \wedge\}$. The algebraic semantics of this logic is studied together with some of its subclasses corresponding to extensions of the above-mentioned expansion. The axiomatic extensions are proved to be *standard complete*.

Keywords BL-algebras, fixed points, storage operators.

1 Introduction

This article collects some unpublished results contained in the author's PhD thesis, written under the guidance of Franco Montagna. Although these findings cannot compare to Franco's remarkable scientific production, they connect in various ways to his research on many-valued logics. For this reason the author wishes to dedicate this small note to his memory, with deep gratitude and the highest esteem.

In [10], Hájek proposed Basic Logic (BL) as a common fragment of all traditional many-valued logics (Łukasiewicz, Gödel and Product Logics). In [5], BL was proved to be complete with respect to the interpretations where the (monoidal) conjunction & and the corresponding implication $\to$, are understood as a continuous t-norm and its residuum, respectively; such a completeness is often referred as *standard completeness*. Although soon generalised by the system MTL (see [9, 13]), Basic Logic remains an important common base for many-valued logics and fuzzy logic.

In [12] two unary operators, L and U, were introduced in the context of BL, to deal with linguistic modifiers like *very, quite*, etc. (see also [11]). The algebraic interpretation of $L(x)$ and $U(x)$ are respectively the "greatest idempotent below" x and the "least idempotent above" x. In [18], Montagna studies a similar operator, called *storage operator*, with the aim of defining *multiplicative quantifiers*. These are

Luca Spada
Dipartimento di Matematica, Università degli Studi di Salerno (Italy).
E-mail: lspada@unisa.it

quantifiers which generalise the t-norm in the same way as standard quantifiers generalise the lattice conjunction and disjunction. Another important feature of the storage operator is that it allows to write a (necessarily infinitary) rule that guarantees the *strong standard completeness* of the related logic [19]. Among other things, the author notices that the storage operator is the maximum fixed point of a formula. Following this hint we introduce an expansion of BL with new connectives which can be interpreted as the (maximum) fixed points of their correspondent formulas. This work can be see as a natural continuation (and generalisation) of the study undertaken in [21, 22, 15].

A peculiar feature of the aforementioned studies on fixed points in many-valued logics is that the existence of fixed points does not hinge on Tarski's fixed point theorem for monotone operators on complete lattices (cf. first order logic with fixed points [7, 1], or the modal μ-calculus [14]). Indeed, in the framework of many-valued logic one can consider formulas which have an interpretation as continuous functions from $[0, 1]^n$ to $[0, 1]$; the existence of their fixed points is then ensured by Brouwer's fixed point theorem. An advantage of such an approach is that, in certain systems like Łukasiewicz logic [22], even negation admits fixed points. This provides for instance with a realisation of the liar's formula $\varphi \leftrightarrow \neg\varphi$, namely $\frac{1}{2}$. Notice that recently an approach to fixed points in Łukasiewicz logic through Tarski's fixed point theorem has also been investigated with interesting applications in computer science [17, 16].

Nevertheless, in the case of BL, the two approaches turn out to be very similar. Indeed, to meet Brouwer's Theorem requirements on continuity one has to restrict to formulas whose interpretation is guaranteed to be continuous. An obvious choice is to restrict the attention to formulas whose only connectives are $\&, \wedge$ and $\vee$ (*continuous formulas*). But such formulas are also guaranteed to have a monotone interpretation, hence Tarski or Brouwer's theorems can be used interchangeably to ensure the existence of their fixed points.

We will show that the maximum fixed point of any continuous formula can be found simply by looking at combinations of meet and joins of formulas of the form $x^n \& a$. Notably, the maximum fixed point of such formulas can be only of two kinds. In particular, the maximum fixed point of formulas of the form $x^n \& a$, for $n > 1$ is the greatest idempotent below a whereas the maximum fixed point of $x \& a$ sits somewhere in between a and the fixed point of $x^n \& a$.

Notice that whereas the operator $L(a)$ of [12] is exactly the storage operator of [18], hence the fixed point of $x^2 \& a$, it is not clear whether $U(a)$ is derivable from the connectives of BL with storage operator. The naïve guess that a *minimum* fixed point might give such an operator is wrong, since the minimum fixed point of formulas such as $x^n \& a$ is 0 for any n and any a.

Finally, regarding other possible extensions of BL, note that adding fixed points to Gödel logic does not increase its expressive power. Indeed the connectives of Gödel logic are either are idempotent or not continuous. Also Product logic does not seem to behave well under this approach: the minimum fixed point is always 0 and the maximum fixed point is given just by the Δ operator. Finally, an expansion of Łukasiewicz logic with fixed points was studied in [22].

The paper is organised as follows. Section 2 recollects some preliminary properties of BL-algebras, this section may be easily skipped by the reader familiar with hoops and ordinal decomposition. In section 3 we introduce the system BL with fixed points. In section 4 we study νBL-algebras and their congruences. In Theorem 3 we prove that all subdirectly irreducible νBL-algebras are linearly ordered. In section 5 we study sub-quasi-varieties of νBL for which the fixed points behave in a simpler way. Finally, in section 6 we prove that all these sub-quasi-varieties are generated by their standard members. Unfortunately we could not prove the same result for the whole quasi-variety of νBL-algebras.

2 Preliminaries

For an updated and extensive account on BL we refer the reader to [6]. Here we briefly recall what is needed in the remainder of the paper. Formulas in Hájek's Basic Logic (BL) are built like in classical logic, with the exception that we now have two *conjunctions*: a strong conjunction & and a lattice conjunction $\wedge$. A complete set of connectives is given by implication $\to$, strong conjunction & and falsum $\bot$. Other connectives can be understood as abbreviations as follows:

$\neg\varphi$	stands for	$\varphi \to \bot$
$\varphi \leftrightarrow \psi$	stands for	$(\varphi \to \psi)\&(\psi \to \varphi)$
$\varphi \wedge \psi$	stands for	$\varphi\&(\varphi \to \psi)$
$\varphi \vee \psi$	stands for	$((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

In this paper we will mainly make use of algebraic methods, we therefore recall some basic facts on the equivalent algebraic semantics of BL.

Definition 1 (Hoop [3]) A *hoop* is a structure $\langle A, \cdot, \Rightarrow, 1\rangle$ such that $\langle A, \cdot, 1\rangle$ is a commutative monoid, and $\Rightarrow$ is a binary operation such that:

$$x \Rightarrow x = 1, \tag{1}$$

$$x \Rightarrow (y \Rightarrow z) = (x \cdot y) \Rightarrow z \tag{2}$$

$$x \cdot (x \Rightarrow y) = y \cdot (y \Rightarrow x). \tag{3}$$

In any hoop, the operation $\Rightarrow$ induces a partial order $\leq$ defined by $x \leq y$ iff $x \Rightarrow y = 1$. Moreover, hoops are precisely the partially ordered commutative integral residuated monoids (*pocrims*) in which the meet operation $\wedge$ is definable by $x \wedge y = x \cdot (x \Rightarrow y)$. Finally, it can be easily checked that hoops satisfy the following *divisibility condition*:

$$\text{If } x \leq y, \text{ then there is an element } z \text{ such that } z \cdot y = x. \tag{4}$$

A hoop is said to be *basic* iff it satisfies

$$(((x \Rightarrow y) \Rightarrow z) \cdot ((y \Rightarrow x) \Rightarrow z)) = 1 \tag{5}$$

A *Wajsberg* hoop is a hoop satisfying:

$$(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x \tag{6}$$

A *cancellative* hoop is a hoop satisfying:

$$x \Rightarrow (x \cdot y) = y \tag{7}$$

A *bounded* hoop is a hoop with an additional constant 0 satisfying the equation $0 \leq x$. A *Wajsberg algebra* is a bounded Wajsberg hoop.

Definition 2 (BL-algebras) BL-algebras are exactly the bounded basic hoops or equivalently the bounded hoops which are isomorphic to subdirect products of linearly ordered bounded hoops.

Definition 3 (Ordinal sum [2]) Let $\langle I, \leq \rangle$ be a totally ordered set with minimum i_0. For all $i \in I$, let W_i be a hoop such that for $i \neq j$, $W_i \cap W_j = \{1\}$, and assume that W_{i_0} is bounded. Then $\bigoplus_{i \in I} W_i$ (the *ordinal sum* of the family $(W_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} W_i$, whose bottom is the minimum of W_{i_0}, whose top is 1, and whose operations are

$$x \Rightarrow y = \begin{cases} x \Rightarrow^{W_i} y & \text{if } x, y \in W_i \\ y & \text{if } \exists i > j \ (x \in W_i \text{ and } y \in W_j) \\ 1 & \text{if } \exists i < j \ (x \in W_i \setminus \{1\} \text{ and } \ y \in W_j) \end{cases}$$

$$x \cdot y = \begin{cases} x \cdot^{W_i} y & \text{if } x, y \in W_i \\ x & \text{if} \, \exists i < j \ (x \in W_i, \, y \in W_j \setminus \{1\}) \\ y & \text{if} \, \exists i < j \ (y \in W_i, \, x \in W_j \setminus \{1\}) \end{cases} .$$

In [2] the following is proved:

Theorem 1 *Every linearly ordered BL-algebra A is the ordinal sum of an indexed family $\langle W_i : i \in I \rangle$ of linearly ordered Wajsberg hoops, where I is a linearly ordered set with minimum i_0, and W_{i_0} is bounded.*

In the sequel, the Wajsberg hoops W_i in Theorem 1 will be called the *Wajsberg components* of A (or just components).

Using the fact that the W_i are closed under hoop operations, it is easy to prove (cf. [2]) that, with reference to Theorem 1, the subalgebras of $A = \bigoplus_{i \in I} W_i$ are those of the form $\mathcal{B} = \bigoplus_{i \in I} U_i$, where for $i \in I$, U_i is a subhoop of W_i (possibly trivial if $i \neq i_0$), and U_{i_0} is a Wajsberg subalgebra of W_{i_0}.

Let $[0,1]_W$ be the *standard* Wajsberg algebra, namely the algebra $\langle [0,1], \Rightarrow_W, \neg_W, 0, 1 \rangle$ where $x \Rightarrow_W y = \min\{1, 1 - x + y\}$ and $\neg_W x = 1 - x$. Let $(\omega)[0,1]_W$ denote the ordinal sum of ω copies of $[0,1]_W$. Then:

Theorem 2 ([2]) *The variety of BL-algebras is generated as a quasivariety by* $(\omega)[0,1]_W$.

3 BL with Fixed Points

We will consider fixed points of formulas only involving $\&, \vee$, and $\wedge$, namely the connectives that have a continuous interpretation in the standard BL-algebras on $[0,1]$. Such formulas will be called *continuous formulas* and the corresponding terms *continuous terms*. We will conventionally write $\varphi(x, y_1, ..., y_n)$ meaning that all propositional variables occurring in φ belong to $\{x, y_1, ..., y_n\}$; the same convention applies to terms and their variables.

Remark 1 Let $f(x, a_1, .., a_n)$ be a continuous map from $[0,1]^{n+1}$ into $[0,1]$. Notice that the function that gives the maximum fixed points of f, as the parameters $a_1, .., a_n$ range in $[0,1]^n$, can have discontinuities. For this reason we will introduce maximum fixed points of continuous formulas statically as a family of new functions, rather than with a single fixed point operator; this approach is also used in modal logic (see e.g., [20]).

Definition 4 The language of ν *Basic Logic* (for short, νBL) is an expansion of BL with a new n-ary connective $\nu p.\varphi(p, q_1, ..., q_n)$ for any continuous BL-formula in $n+1$ propositional variables $\varphi(p, q_1, ..., q_n)$. The system νBL is axiomatised by all axioms and rules of BL (see e.g. [10, Definition 2.2.4] or [6, Vol. 1, pag. 355]) plus the following ones:

1. $\varphi(\nu p.\varphi(p, \bar{q}), \bar{q}) \leftrightarrow \nu p.\varphi(p, \bar{q})$,
2. If $[\varphi(\psi, \bar{q}) \leftrightarrow \psi]$ then $\psi \to \nu p.\varphi(p, \bar{q})$,
3. $\big(\nu p.(p^2 \& \psi)\big) \vee \big(\nu p.(p^2 \& \xi)\big) \leftrightarrow \nu p.(p^2 \& (\psi \vee \xi))$,
4. $(\nu p.\psi(p, r)) \& (\nu p.\psi(p, s)) \to \nu p.(\psi(p, r \& s))$;

here $\bar{q}$ stands for a tuple of propositional variables of suitable arity.

Henceforth, we write x^n for the n-time product $x \cdot .... \cdot x$, we conventionally set $x^0 = 1$. It is trivial to see that the equivalent algebraic semantics of νBL is given by the following quasi-variety of structures.

Definition 5 (νBL-algebras) Let $\mathcal{C}$ be the set of *continuous terms* i.e., terms in the language of BL-algebras in which only the connectives $\cdot$, $\vee$ and $\wedge$ appear. A *νBL-algebra* is a structure

$$\langle A, \cdot, \Rightarrow, \vee, \wedge, 0, 1, \{\nu x.t(x, \bar{y})\}_{t(x,\bar{y}) \in \mathcal{C}} \rangle$$

such that $\langle A, \cdot, \Rightarrow, \vee, \wedge, 0, 1\rangle$ is a BL-algebra, and the following equations hold:

$$t(\nu x.t(x, \bar{y}), \bar{y}) = \nu x.t(x, \bar{y}), \tag{νBL1}$$

$$\text{If } [t(u, \bar{y}) = u] \text{ then } u \leq \nu x.t(x, \bar{y}), \tag{νBL2}$$

$$(\nu x.x^2 \cdot y) \vee (\nu x.x^2 \cdot z) = \nu x.(x^2 \cdot (y \vee z)), \tag{νBL3}$$

$$(\nu x.t(x, y)) \cdot (\nu x.t(x, z)) \leq \nu x.(t(x, y \cdot z)). \tag{νBL4}$$

It is understood in the notation that the variable x is bound in $\nu x.t(x, \bar{y})$, and the usual rules for substitutions in presence of bindings apply.

Example 1 If $*$ is a continuous t-norm and $\Rightarrow$ is its residuum, $\langle [0,1], *, \Rightarrow, 0, 1\rangle$ is a BL-algebra; the derived operations $\wedge$ and $\vee$ coincide here with minimum and maximum between two numbers. Such algebras are often referred as the *standard BL-algebras*. The interpretations of continuous terms in these algebras contain only the operations $*, \wedge$ and $\vee$, which are obviously continuous w.r.t. the Euclidean topology on [0,1]. So, if $t(x, y_1, ..., y_n)$ is a continuous term, its interpretation $\tilde{t}$ in a standard BL-algebra is a continuous function from $[0,1]^{n+1}$ into $[0,1]$. Under these assumptions the term $\nu x.t(x, y_1, ..., y_n)$ is a function from $[0,1]^n$ into $[0,1]$ obtained as follows. For any tuple $a_1, ..., a_n \in [0,1]$ the function $\tilde{t}(x, a_1, ..., a_n)$ is a continuous function from [0,1] into [0,1], so by Brouwer's theorem it has at least a fixed point in, hence it has a maximum fixed point a; we set $\widetilde{\nu x.t}(a_1, ..., a_n) = a$. By varying the tuple $a_1, ..., a_n$ in $[0,1]$, we obtain the whole interpretation of $\widetilde{\nu x.t}$.

To understand the intuitive meaning of the above axioms one should look at the term $t(x, y_1, ..., y_n)$ as a function from A^{n+1} to A. So, $\nu x.t(x, y_1, ..., y_n)$ is a n-ary function. Axiom (νBL1) then states that $\nu x.t(x, a_1, ..., a_n)$ is a solution to the equation $t(x, \bar{a}) = x$. Axiom (νBL2) states that it is the largest such. Finally, axioms (νBL3) and (νBL4) are technical requirements that guarantee that νBL-algebras are subdirect products of linearly ordered ones. While [18, p. 308] contains an example showing that (νBL3) is necessary if one wants to work with a pre-linear class of algebras, we could not find an example showing that (νBL4) is not redundant.

4 General properties of νBL-algebras

Lemma 1 *For every νBL-algebra and every $a \in A$ the element given by $\nu x.(x^2 \cdot a)$ is the largest idempotent below a.*

Proof. We start by noticing that the solutions to the equation $x^2 \cdot a = x$ are exactly all idempotents below a. Indeed, if e is such a solution, then since in any BL-algebra $x \cdot y \leq x, y$, we have $e \geq e^2 \geq e^2 \cdot a = e$, so e is idempotent. Moreover $a \geq e^2 \cdot a = e$, so e is below a. Vice versa, if e is an idempotent below a, then $e^2 \cdot a = e \cdot a \geq e \cdot e = e$; where the last inequality holds because in any BL-algebra $\cdot$ is increasing in both coordinates. In addition, $e^2 \cdot a \leq e$, so combining the two last inequalities we obtain $e^2 \cdot a = e$ and the first claim is proved.

By axiom (νBL1), $\nu x.(x^2 \cdot a)$ is a solution to the equation $x^2 \cdot a = x$, so by the above reasoning it is idempotent. Further by axiom (νBL2), it is the maximum among the idempotents below a and the lemma is proved. □

We note that, whereas in [18] the author needs to assume the existence of the maximum idempotent below any element in the definition of a *weakly saturated* BL-algebra, with our approach this is not necessary as those elements must exist, being the images of functions of the algebra.

Corollary 1 *In every νBL-algebra one has $\nu x.(x^2 \cdot a) = \nu x.(x^2 \cdot (\nu x.(x^2 \cdot a)))$*

Proof. By Lemma 1 $\nu x.(x^2 \cdot a)$ is an idempotent so $\nu x.(x^2 \cdot (\nu x.(x^2 \cdot a)))$ is the largest idempotent below $\nu x.(x^2 \cdot a)$, which is $\nu x.(x^2 \cdot a)$ itself. □

In the light of Remark 1, the second term in the statement of Corollary 1 seems not admissible, as it presents nested occurrences of νx. However, the term can be obtained by a substitution of y for $\nu x.(x^2 \cdot a)$ in the term $\nu x.x^2 \cdot y$.

Corollary 2 *Every term in the language of BL-algebras with storage operator can be faithfully translated in a term of νBL-algebras.*

Proof. To translate a term in the language of BL-algebras with storage operators in a νBL term we only have to substitute all the occurrences of the storage operator applied to a certain variable x by the νBL term $\nu y.(y^2 \cdot x)$, Lemma 1 ensures that such a substitution gives an equivalent term. □

Lemma 2 *For every νBL-algebra A and $a \in A$, $\nu x.(x^2 \cdot a) = \nu x.(x^n \cdot a)$, whenever $n \geq 2$.*

Proof. Let $d = \nu x.(x^2 \cdot a)$, then by Lemma 1 $d = d^2 = d^n$, so $d = d^2 \cdot a = d^n \cdot a$. Hence by ($\nu$BL2) $\nu x.(x^2 \cdot a) \leq \nu x.(x^n \cdot a)$. For the other inequality notice that, letting $e = \nu x.(x^n \cdot a)$, we have $e = e^n \cdot a \leq e^n \leq e^2 \leq e$, so e is an idempotent below a. But by Lemma 1, d is the largest such, hence $\nu x.(x^2 \cdot a) \geq \nu x.(x^n \cdot a)$ □

Lemma 3 *For any $a \leq b$ in a νBL algebra A, $\nu x.(x^2 \cdot a)$ is also a solution for the fixed point equation $x^2 \cdot b = x$.*

Proof. Obviously $(\nu x.(x^2 \cdot a))^2 \cdot b \leq \nu x.(x^2 \cdot a)$, but for the monotonicity of $\cdot$, $\nu x.(x^2 \cdot a) = (\nu x.(x^2 \cdot a))^2 \cdot a \leq (\nu x.(x^2 \cdot a))^2 \cdot b$ and the claim is proved. □

Lemma 4 *For every $m \in \mathbb{N}$,*

$$\nu x.(x^2 \cdot a_1 \cdot ... \cdot a_n) = \nu x.(x^2 \cdot a_1^m \cdot ... \cdot a_n^m).$$

Proof. Set $b := \nu x.(x^2 \cdot a_1 \cdot ... \cdot a_n)$, then we have that

$$\begin{aligned} b^2 \cdot a_1^m \cdot ... \cdot a_n^m &= (b^2 \cdot a_1 \cdot ... \cdot a_n) \cdot a_1^{m-1} \cdot ... \cdot a_n^{m-1} = \\ &= b^2 \cdot a_1^{m-1} \cdot ... \cdot a_n^{m-1} = ... = \\ &= b. \end{aligned}$$

where the second and fourth equalities hold because b is a fixed point of that term and it is idempotent. So we conclude by axiom (νBL2), that $b \leq \nu x.(x^2 \cdot a_1^m \cdot ... \cdot a_n^m)$. On the other hand $a_1^m \cdot ... \cdot a_n^m \leq a_1 \cdot ... \cdot a_n$, so by Lemma 3 $\nu x.(x^2 \cdot a)_1^m \cdot ... \cdot a_n^m$ is a fixed point for $x^2 \cdot a_1 \cdot ... \cdot a_n$ and so, again by (νBL2), $\nu x.(x^2 \cdot a)_1^m \cdot ... \cdot a_n^m \leq b$ and the claim is proved. □

Remark 2 Notice that since the variety of BL-algebras is generated by its linearly ordered members, each continuous term $t(x, y_1, ..., y_n)$ is equivalent over BL to one of the form

$$\bigvee_{i \in I} \bigwedge_{j \in J} \big(x^{k_{0ij}} \cdot y_1^{k_{1ij}} \cdot ... \cdot y_n^{k_{nij}}\big). \tag{8}$$

Lemma 5 *Let A be a νBL-algebra and $a_1, ..., a_n \in A$. Then for every continuous term $t(x, y_1, ..., y_n)$ one has that $\nu x.(x^2 \cdot a_1 \cdot ... \cdot a_n) \leq \nu x.t(x, a_1, ..., a_n)$.*

Proof. Recall that by Remark 2, $t(x, y_1, ..., y_n)$ can be thought to be of the form

$$\bigvee_{i \in I} \bigwedge_{j \in J} \big(x^{k_{0ij}} \cdot y_1^{k_{1ij}} \cdot ... \cdot y_n^{k_{nij}}\big).$$

Let m be the maximum among all k_{lij}, for $l \leq n$, $i \in I$ and $j \in J$. Then

$$a_1^m \cdot ... a_n^m \leq a_1^{k_{1ij}} \cdot ... \cdot a_n^{k_{nij}}$$

for all $l \leq n$, $i \in I$ and $j \in J$, so by Lemmas 2 and 3 and

$$t\big((\nu x.(x^2 \cdot a)_1^m \cdot ... a_n^m), a_1, ..., a_n\big) = \nu x.(x^2 \cdot a)_1^m \cdot ... a_n^m.$$

But, by Lemma 4, $\nu x.(x^2 \cdot a)_1^m \cdot ... a_n^m = \nu x.(x^2 \cdot a)_1 \cdot ... a_n$, so by axiom ($\nu$BL2), $\nu x.(x^2 \cdot a_1 \cdot ... \cdot a_n) \leq \nu x.t(x, a_1, ..., a_n)$. □

Let us call νBL^+ the variety generated by νBL-algebras. Obviously all equations true in νBL are also true in νBL^+. In order to characterise subdirectly irreducible νBL^+-algebras, we study congruences in νBL^+-algebras and the associated νBL-filters.

Definition 6 If A is a νBL^+, we call $F \subseteq A$ a νBL-filter if it is a filter of the BL-reduct and has the following closure property:

$$\text{If } a_1, ..., a_n \in F \text{ then } \nu x.(x^2 \cdot a_1 \cdot ... \cdot a_n) \in F. \tag{9}$$

Recall that in BL-algebras there is a bijective correspondence between filters and congruences given by the association:

$$\theta \mapsto F_\theta := \{a \in A \mid a\theta 1\} \tag{10}$$

$$F \mapsto \theta_F := \{(a,b) \mid a \Rightarrow b \in F \text{ and } b \Rightarrow a \in F\}. \tag{11}$$

Such a correspondence extends to νBL-filters and congruences.

Lemma 6 *Let A be a νBL^+-algebra. There is a bijective correspondence between νBL-filters and congruences on A given by* (10) *and* (11).

Proof. If θ is a νBL-congruence, then F_θ is obviously a BL-filter. To prove (9), suppose that $a_1\theta 1, ..., a_n\theta 1$, then for any $i \leq n$, $\big(\nu x.(x^2 \cdot a_i)\big)\theta\big(\nu x.x^2 \cdot 1\big)$, i.e. $\big(\nu x.(x^2 \cdot a_i)\big)\theta 1$. So,

$$(\nu x.(x^2 \cdot a_1)) \in F, ..., (\nu x.(x^2 \cdot a_n)) \in F.$$

Hence

$$(\nu x.(x^2 \cdot a_1)) \cdot ... \cdot (\nu x.(x^2 \cdot a_n)) \in F,$$

so, by (νBL4)

$$(\nu x.(x^2 \cdot a_1 \cdot ... \cdot a_n)) \in F.$$

Vice versa, if F is a ν-BL filter, we already know that θ_F is a BL-congruence. To see that indeed it is a νBL-congruence, suppose that $a_1\theta b_1, ..., a_n\theta b_n$. Then by definition $a_1 \Rightarrow b_1, ..., a_n \Rightarrow b_n \in F$. So by (9), for $i \leq n$, $\nu x.x^2 \cdot (a_i \Rightarrow b_i) \in F$. By Lemma 5, this implies that for any continuous term t, $\nu x.t(x, a_i \Rightarrow b_i) \in F$. Finally, by ($\nu$BL4) $\nu x.t(x,a) \cdot \nu x.t(x, a \Rightarrow b) \leq \nu x.t(x, a \cdot (a \Rightarrow b)) \leq \nu x.t(x,b)$, so by residuation

$$\nu x.t(x, a \Rightarrow b) \leq \nu x.t(x,a) \Rightarrow \nu x.t(x,b).$$

So, $\nu x.t(x,a_i) \Rightarrow \nu x.t(x,b_i) \in F$. Similarly one can prove that $\nu x.t(x,b_i) \Rightarrow \nu x.t(x,a_i) \in F$, hence θ is a νBL-congruence. $\square$

Lemma 7 *Let A be a νBL^+-algebra and $a \in A$. The νBL-filter generated by a, F_a is given by the set $\{x \in A \mid x \geq \nu x.(x^2 \cdot a)\}$.*

Proof. Call the above set $F(a)$. Clearly $\nu x.(x^2 \cdot a) \in F_a$ and since a νBL-filter is upward closed $F(a) \subseteq F_a$. For the other inclusion we only have to prove that $F(a)$ is a νBL-filter, as F_a is the minimal filter containing a. Clearly $1 \in F(a)$. If $x, x \Rightarrow y \in F(a)$ then

$$\nu x.(x^2 \cdot a) = (\nu x.(x^2 \cdot a)) \cdot (\nu x.(x^2 \cdot a)) \leq x \cdot (x \Rightarrow y) \leq y,$$

hence $y \in F(a)$. Finally if $z \in F(a)$, then by Corollary 1

$$\nu x.x^2 \cdot z \geq \nu x.x^2 \cdot (\nu x.(x^2 \cdot a)) = \nu x.(x^2 \cdot a),$$

so $\nu x.x^2 \cdot z \in F(a)$ $\square$

Lemma 8 *Every subdirectly irreducible νBL^+-algebra is linearly ordered.*

Proof. If A is subdirectly irreducible νBL$^+$-algebra then it has a minimal non-trivial congruence. The νBL-filter associated to this congruence by Lemma 6 is minimal and non-trivial as well, hence it is generated by some $c \in A$ different from 1. Let us indicate this filter by F_c. Suppose that there exist $a, b \in A$ such that neither $a \leq b$ or $b \leq a$, then $F_{a\Rightarrow b}$ and $F_{b\Rightarrow a}$ are both non-trivial filters, so they both contain F_c. This implies, by Lemma 7, that $c \geq \nu x.(x^2 \cdot (a \Rightarrow b))$ and $c \geq \nu x.(x^2 \cdot (b \Rightarrow a))$, which gives $c \geq [\nu x.(x^2 \cdot (a \Rightarrow b))] \vee [\nu x.(x^2 \cdot (b \Rightarrow a))]$. By axiom ($\nu$BL3) of Definition 5, this implies $c \geq [\nu x.(x^2 \cdot (a \Rightarrow b) \vee (b \Rightarrow a))] = \nu x.(x^2 \cdot 1) = 1$, a contradiction. □

Theorem 3 *Every νBL-algebras is the subdirect product of linearly ordered νBL$^+$-algebras.*

Proof. By Birkhoff's subdirect representation theorem any νBL-algebra A is the subdirect product of subdirectly irreducible algebras and they all belong to νBL$^+$. Hence, by Lemma 8, A is the subdirect product of linearly ordered νBL$^+$-algebras. □

Lemma 9 *Every term of the form $\nu x.t(x)$ is equivalent to*

$$\bigwedge \bigvee_{i \in I} \nu x.(x^{n_i} \cdot a_i)$$

where, conventionally, we write $\nu x.x^0 \cdot a = a$. In other words in every continuous term the functions νx can be pushed inside until the basic parts of the form $x^n \cdot a$.

Proof. Since every continuous term is equivalent to a formula $\bigwedge \bigvee_{i \in I}(x^{n_i} \cdot a_i)$ we only have to prove that νx commutes with $\wedge$ and $\vee$. But in every linearly ordered νBL$^+$-algebra the following equations hold

$$\nu x. \left(t_1(x) \vee t_2(x)\right) = \nu x.t_1(x) \vee \nu x.t_2(x)$$

and

$$\nu x. \left(t_1(x) \wedge t_2(x)\right) = \nu x.t_1(x) \wedge \nu x.t_2(x).$$

Hence, by Theorem 3 they hold in every νBL-algebra. □

In [18], the storage operator of a is characterised as the largest solution to the equation $x^2 \cdot a = x$. In Lemma 2 we showed that it is also the largest solution to the equation $x^n \cdot a = x$, for any $n \geq 2$. So the only fixed points which are not definable trough the storage operator are the ones of the form $\nu x.x \cdot a$. The following remark, combined with Theorem 2, deceptively suggests that the two fixed points coincide.

Remark 3 For any a in the BL-algebra $(\omega)[0,1]_W$, one has $\nu x.(x^2 \cdot a) = \nu x.(x \cdot a)$. Indeed, $x \cdot y$ is defined in $[0,1]_W$ as $\max\{x + y - 1, 0\}$, so $[0,1]_W$ has only two idempotent elements which are 0 and 1. An easy calculation shows that the maximum solution to the equation $x \cdot a = x$ is 1 if $a = 1$ and 0 otherwise. But then in $(\omega)[0,1]_W$, $\nu x.(x \cdot a)$ is the greatest idempotent below a, whence $\nu x.(x^2 \cdot a) = \nu x.(x \cdot a)$.

Unfortunately although this result holds for the "generic" algebra $(\omega)[0,1]_W$, it does not hold in general, as next example proves. This proves also that fixed points are not equationally definable in BL-algebras.

Example 2 Consider the BL-algebra $\mathcal{CH} = \mathcal{C} \oplus \mathbb{Z}^-$ where $\mathcal{C}$ is the Chang's MV-algebra (see for instance [4] for details) and $\mathbb{Z}^-$ is the negative cone of the integer numbers. Take $a \in \mathbb{Z}^-$, then since there are no idempotents in $\mathbb{Z}^-$, but 0 and 1, $\nu x.(x^2 \cdot a) = 0$ if, and only if, $a \neq 1$. But the maximum fixed point of $x \cdot a$ is the maximum of $\mathcal{C}$. So $\mathcal{CH}$ can be expanded to a νBL algebra in which $\nu x.(x \cdot a) \neq \nu x.(x^2 \cdot a)$.

5 Subclasses of νBL

Since the only relevant fixed points are $\nu x.(x \cdot a)$ and $\nu x.(x^2 \cdot a)$, it makes sense to further investigate their properties.

Let us define $\Box(a) = \nu x.(x^2 \cdot a)$ and $\triangle(a) = \nu x.(x \cdot a)$. Let us also use the following shorthands:

$$\triangle^n(a) = \triangle(\triangle^{n-1}(a)),$$
$$\triangle(a)^n = \underbrace{\triangle(a) \cdot \ldots \cdot \triangle(a)}_{n \text{ times}},$$

Lemma 10
1. *For every $n \geq 1$, $\triangle(a)^n \geq \Box(a)$,*
2. *For every $n \geq 1$, $\triangle^n(a) \geq \Box(a)$ and $\Box(\triangle^n(a)) = \Box(a)$,*
3. *If $m \leq n$ then $\triangle^m(a) \geq \triangle^n(a)$ and $\triangle(a)^m \geq \triangle(a)^n$,*
4. *For any $n \in \mathbb{N} \setminus \{0\}$ $\triangle(a)^n \geq \triangle^2(a)$.*

Proof.

1. The element $\Box(a)$ is idempotent hence $\Box(a) = \Box(a)^2 \cdot a = \Box(a) \cdot a$, so $\Box(a)$ is a solution to the equation $x \cdot a = x$ but by definition $\triangle(a)$ is the greatest one, so $\triangle(a) \geq \Box(a)$. But then $\triangle(a)^n \geq \Box(a)^n = \Box(a)$.
2. By induction on n. One part of the basic step, namely that $\triangle(a) \geq \Box(a)$, is proved in the claim above; furthermore $\Box(\triangle(a)) = \Box(a)$ is true because $\Box(a) \leq \triangle(a) \leq a$, whence the largest idempotent below a is also the largest idempotent below $\triangle(a)$.
   For the inductive step suppose that $\triangle^n(a) \geq \Box(a)$ and $\Box(\triangle^n(a)) = \Box(a)$, then $\Box(\triangle^n(a)) \cdot \triangle^n(a) = \Box(\triangle^n(a))^2 \cdot \triangle^n(a) = \Box(\triangle^n(a))$. Hence $\Box(\triangle^n(a)) \leq \triangle^{n+1}(a)$, but, by induction hypothesis $\Box(\triangle^n(a)) = \Box(a)$. Finally $\Box(\triangle^{n+1}(a)) = \Box(a)$ because $a \geq \triangle^{n+1}(a)) \geq \Box(a)$.
3. We have that

$$\begin{aligned}\triangle^n(a) &= \triangle^n(a) \cdot \triangle^{n-1}(a) = \\ &= \triangle^n(a) \cdot \triangle^{n-1}(a) \cdot \triangle^{n-2}(a) = \ldots = \\ &= \triangle^n(a) \cdot \triangle^{n-1}(a) \cdot \ldots \cdot \triangle^m(a) \leq \triangle^m(a).\end{aligned}$$

   The second claim is obvious.

4. By induction on n. The basic step holds because of item 3 above.
   If $\triangle(a)^n \geq \triangle^2(a)$ then $\triangle(a)^{n+1} \geq \triangle^2(a) \cdot \triangle(a) = \triangle^2(a)$.

$\square$

Proposition 1 *Let A be a νBL-algebra, then for every $a \in A$, the following are equivalent:*

$$\exists n > 1(\triangle(a)^n = \triangle(a)), \tag{12}$$

$$\exists n > 1(\triangle^n(a) = \triangle(a)), \tag{13}$$

$$\triangle(a) = \square(a). \tag{14}$$

Proof. (12) implies (13). We have that:

$$\triangle(a) = \triangle(a)^n \leq ... \leq \triangle(a)^2 \leq \triangle(a),$$

whence $\triangle(a) = \triangle(a) \cdot \triangle(a)$, so $\triangle(a) \leq \triangle^2(a)$. Since by Lemma 10 item 2, $\triangle(a) \geq \triangle^2(a)$, hence they must be equal.
(13) implies (14). Consider the following chain of implications:

$$\triangle^n(a) = \triangle(a) \text{ implies } \triangle(\triangle^{n-1}(a)) = \triangle(a) \text{ implies } \triangle^{n-1}(a) \cdot \triangle(a) = \triangle(a)$$
$$\text{implies } \triangle^{n-1}(a) \geq \triangle^{n-1}(a) \cdot \triangle(a) = \triangle(a) \text{ implies } \triangle^{n-1}(a) = \triangle(a).$$

Then we get that $\triangle^2(a) = \triangle(a)$, hence $\triangle(a)$ is an idempotent and since $a \geq \triangle(a) \geq \square(a)$ it must be $\triangle(a) = \square(a)$, because $\square(a)$ is the largest idempotent below a. Finally, that (14) implies (12) is an immediate consequence of Lemma 1.
$\square$

Definition 7 An *archimedean* νBL-algebra is an algebra in which, for any element a, one of the equivalent conditions of Proposition 1 holds.

Archimedean νBL-algebras are a proper sub-quasi-variety of νBL-algebras, as can be seen from Example 2.

The properties listed in Proposition 1 do not completely characterise the behavior of $\triangle$ w.r.t. $\square$. A comparison between the algebra introduced in Example 2 and the one on next example, can be instructive at this point.

Example 3 Let $\mathbb{Z}^-$ be as in Example 2 and let us call $(\omega)\mathbb{Z}^-$ the ordinal sum of the standard Wajsberg algebra $[0,1]_W$ and ω copies of the hoop $\mathbb{Z}^-$. Let us take a in the highest component of the ordinal sum. Since there are no idempotents in $(\omega)\mathbb{Z}^-$, but 0 and 1, $\nu x.(x^2 \cdot a) = 0$ if, and only if, $a \neq 1$. Notice that given $b \neq 1$ in a component different from the first $\nu x.x \cdot b$ is the maximum of the previous component. From this we deduce that for any $n > 0$, $\triangle^n(a) \neq \square(a)$.

So another interesting (proper) subclass of νBL-algebras is given by those members in which there exists $n > 0$ for which $\triangle^n(a) = \triangle^{n-1}(a)$. We will see that also in this case $\triangle(a)$ and $\square(a)$ are related.

Lemma 11 *For any element a of a νBL-algebra the following hold:*

1. $\exists n(\triangle^n(a) = \triangle^{n-1}(a))$ *if, and only if,* $\exists m(\triangle^m(a) = \square(a))$*;*

2. $\exists n(\triangle(a)^n = \triangle(a)^{n-1})$ *if, and only if,* $\exists m(\triangle(a)^m = \square(a))$*;*

Proof.

1. One direction is obvious. For the other one notice that if $\triangle^n(a) = \triangle^{n-1}(a))$, then $\triangle^{n-1}(a) = \triangle(\triangle^{n-1}(a)) = \triangle^{n-1}(a) \cdot \triangle^n(a) = \triangle^{n-1}(a) \cdot \triangle^{n-1}(a)$. But $\square(a)$ is the largest idempotent below a, hence $\triangle^{n-1}(a) = \square(a)$.
2. For the left-to-right implication consider that $\triangle(a)^{2n} = \triangle(a)^{n-1} \cdot \triangle(a)^{n-1} = \triangle(a)^n \cdot \triangle(a)^{n-2} = \triangle(a)^{n-1} \cdot \triangle(a)^{n-2} = ... = \triangle(a)^n$. Hence $\triangle(a)^n = \square(a)$. For the other direction just notice that $\triangle(a)^m \geq \triangle(a)^{m+1} \geq \triangle(a)^{2m} = \triangle(a)^m$.

$\square$

Note that by Lemma 10 item 4, the two parts of condition 2 of the proposition above are stronger than the two parts of condition 1. To see that the two conditions are not equivalent one may consider the algebra of the Example 2, where, with a as in the example, there exists n such that $\triangle^n(a) = \square(a)$ but for no m one has that $\triangle(a)^m = \square(a)$.

Definition 8

– A νBL-algebra is said ^{m}*-archimedean* if it satisfies $\triangle(a)^m = \square(a)$.
– A νBL-algebra is said ${}_m$*-archimedean* if it satisfies $\triangle^m(a) = \square(a)$.
– A νBL-algebra is said ${}_\infty$*-archimedean* if it satisfies

$$\exists m > 1(\triangle^m(a) = \square(a)).$$

– A νBL-algebra is said ${}^\infty$*-archimedean* if it satisfies

$$\exists m > 1(\triangle^m(a) = \square(a)).$$

Notice that archimedean, ${}_m$-archimedean and ^{m}-archimedean νBL-algebras are equationally definable subclasses of νBL-algebras, hence sub-quasi-varieties. ${}_\infty$-archimedean and ${}^\infty$-archimedean can be seen as unions, for $m \in \mathbb{N}$, of the quasi-varieties of ${}_m$-archimedean and ^{m}-archimedean, respectively.

Figure 1 should make clear their reciprocal relationships.

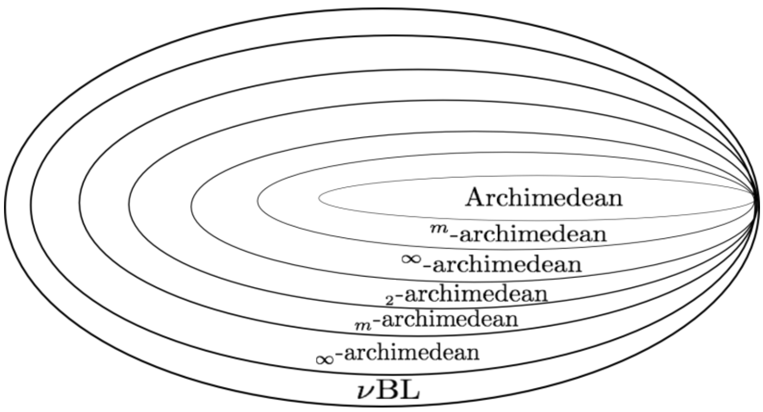

Fig. 1 Subclasses of the variety of νBL-algebras

6 Standard Completeness

Although we are not able to give a proof of the standard completeness of the logic νBL, all the other extensions introduced in the previous section are standard complete. The key remark is that if A is an algebra in one of the subclasses above then the following holds:

$$\forall a \in A \ \exists n \text{ such that for every } u, v \geq n \ \triangle^u(a) = \triangle^v(a). \tag{15}$$

This implies that when considering a finitely generated νBL-algebra we can assume without loss of generality that the set of generators is already closed under $\triangle$ and $\Box$. This property will allow us to embed every finitely generated νBL-algebra satisfying (15) in a standard νBL-algebra.

The proof of the following lemma follows the lines of [18].

Lemma 12 *Every finitely generated subdirectly irreducible $_\infty$-archimedean νBL^+-algebra is the ordinal sum of its Wajsberg components containing its generators.*

Proof. Let A be a finitely generated subdirectly irreducible νBL$^+$-algebra, whose set of generators is $G = \{g_1, ..., g_n\}$. Without loss of generality we may suppose $0 \in G$ and that if $g \in G$ then $\Box(g), \triangle(g) \in G$. Since $\Box(\Box(g)) = \Box(g)$ and for some m, $\triangle^m(g) = \triangle^{m+1}(g)$, this does not break the finiteness of G.

Let $W_1, ..., W_n$ be the (non necessary distinct) Wajsberg components to which every $g \in G$ belongs. Since A is finitely generated, every $b \in A$ is the interpretation of some term all of whose variables are in G. We prove by induction on the complexity of this term that $b \in \bigcup_{i \leq n} W_i$.

Since 0 is in G and 1 belongs to every component, the basic step holds. Concerning the inductive cases of $\cdot$ and $\Rightarrow$, the statement directly comes from the definition of ordinal sum, indeed if $a_i \in W_i$ and $a_j \in W_j$ then $a_i \cdot a_j, a_i \Rightarrow a_j \in W_i \cup W_j$.

As regards the case of ν, since we are in a $_\infty$-archimedean algebra, it is enough to check that if a belongs to some W_i then, for a suitable m, $\triangle^m(a)$ and $\Box(a)$ are in $\bigcup_{j \leq n} W_j$. Note that if a and b are in the same component then if $\Box(a)$ does not belong to the same component it must be equal to $\Box(b)$. This also holds for $\triangle(a)$ and $\triangle(b)$, for if $\triangle(a)$ does not belong to the same component of a it must be the maximum of the previous component and the same holds for b. But then it is sufficient to take $\Box(g_i)$ (resp. $\triangle^m(g_i)$) where g_i is the generator which belongs to the same component of a. As $\Box(g_i)$ (resp. $\triangle^m(g_i)$) is in W_i by hypothesis the claim is proved. $\Box$

Theorem 4 *Every finitely generated subdirectly irreducible $_\infty$-archimedean νBL^+-algebra can be embedded in a standard one.*

Proof. Let A be such an algebra, by Lemma 8 A is linearly ordered. Let $A = \bigoplus_{i \leq n} A_i$ be as given by Lemma 12. Then, for all $i \leq n$, A_i is either a Wajsberg algebra or a cancellative hoop whose previous component is the two elements algebra. In the first case, by Di Nola's theorem [8] A_i embeds in an ultrapower of the Łukasiewicz standard algebra. Otherwise $A_{i-1} \oplus A_i$ embeds in an ultrapower of the standard product algebra.

Let us now build a standard νBL-algebra. We start substituting every Wajsberg algebra with the standard Łukasiewicz algebra, and every ordinal sum $W_{i-1} \oplus W_i$, where W_i is cancellative and W_{i-1} is the two-elements hoop, with the standard product algebra; the structure obtained is a νBL-algebra (where the ν-functions are defined in the obvious way).

As a second step we replace every standard algebra by an isomorphic copy having domain $[\frac{i-1}{k}, \frac{i}{k}]$. What we obtain is a standard νBL-algebra which is isomorphic to the previous. We only need to prove that A embeds into some of its ultrapowers.

In [2] it is proved that if every basic hoop H_i embeds in an ultrapower of a basic hoop $\mathcal{K}_i$, then the ordinal sum $\bigoplus_{i \in I} H_i$ embeds in an ultrapower of $\bigoplus_{i \in I} \mathcal{K}_i$. Hence in this case we have that the BL reduct A embeds in the BL reduct of the standard algebra we just constructed, call Ψ such an embedding.

It remains to prove that such an embedding preserves fixed points, namely that $\Psi(\nu x.t(x, y_1, ..., y_n)) = \nu x.(t(x, \Psi(y_1), ..., \Psi(y_n)))$. But this directly comes from the fact that Ψ preserves the order of the components. □

Corollary 3 *The quasi-varieties of archimedean, $_m$-archimedean and ^{m}-archimedean νBL-algebras are generated by their standard members.*

Proof. If a quasi-equation q fails in the quasi-variety of archimedean νBL-algebras, then it fails in some subdirectly irreducible archimedean $\nu\mathrm{BL}^+$-algebra. But quasi-equations are preserved by taking subalgebras, so by Theorem 4, q fails in a standard algebra, and the claim is proved. □

References

1. P. Aczel. An introduction to inductive definition. In J. Barwise, editor, *Handbook of Mathematical Logic*, volume 90 of Studies in Logic, pages 739–782. North-Holland, 1977.
2. P. Aglianò and F. Montagna. Varieties of BL algebras I: general properties. *Journal of Pure and Applied Algebra*, 181:105–129, 2003.
3. W. J. Blok and I. M. A. Ferreirim. On the structure of hoops. *Algebra Universalis*, 43:233–257, 2000.
4. R. Cignoli, I. D'Ottaviano, and D. Mundici. *Algebraic Foundations of Many-valued Reasoning*, volume 7 of *Trends in Logic, Studia Logica Library.* Kluwer Academic, 2000.
5. R. Cignoli, F. Esteva, L. Godo, and A. Torrens. Basic fuzzy logic is the logic of t-norms and their residua. *Soft Computing*, 4:106–112, 2000.
6. P. Cintula, P. Hàjek, and C. Noguera, editors. *Handbook of Mathematical Fuzzy Logic (in 2 volumes)*, volume 37 and 38 of *Studies in Logic, Mathematical Logic and Foundations.* College Publications, London, 2011.
7. A. Dawar and Y. Gurevich. Fixed points logics. *The Bulletin of Symbolic Logic*, 8:65–90, 2002.
8. A. Di Nola. Representation and reticulation by quotients of MV-algebras. *Ricerche di Matematica*, XL(2):291–297, 1991.
9. F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets Systems*, 124:271–288, 2001.
10. P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic, Studia Logica Library.* Kluwer Academic, Berlin, 1998.
11. P. Hájek. On very true. *Fuzzy Sets Systems*, 124:329–333, 2001.
12. P. Hájek. Some hedges for continuous t-norm logics. *Neural Networks World*, 2:159–164, 2002. Preprint.
13. S. Jenei and F. Montagna. A proof of standard completeness of Esteva and Godo's logic MTL. *Studia Logica*, 70:183–192, 2002.
14. D. Kozen. Results on the propositional mu-calculus. *Theoretical Computer Science*, 27:333–354, 1983.

15. E. Marchioni and L. Spada. Advances in ŁΠ logic with fixed points. *Logic Journal of the IGPL*, 19(3):476–489, 2011.
16. M. Mio. Upper-expectation bisimilarity and Łukasiewicz μ-calculus. In *Foundations of Software Science and Computation Structures*, pages 335–350. Springer, 2014.
17. M. Mio and A. Simpson. Łukasiewicz μ-calculus. *EPTCS 126*, page 87, 2013.
18. F. Montagna. Storage operators and multiplicative quantifiers in many valued logics. *Journal of Logic and Computation*, 14(2), 2004.
19. F. Montagna. Notes on strong completeness in Łukasiewicz, product and BL logics and in their first-order extensions. In S. A. et. al., editor, *Algebraic and Proof-theoretic aspects*, volume 4460 of *Lectures Notes in Artificial Intelligence*, pages 247–274. Springer-Verlag, 2007.
20. L. Santocanale and Y. Venema. Completeness for flat modal fixpoint logics. *Annals of Pure and Applied Logic*, 162(1):55–82, 2010.
21. L. Spada. ŁΠ logic with fixed points. *Archive for Mathematical Logic*, 47(7-8):741–763, 2008.
22. L. Spada. μMV-algebras: an approach to fixed points in Łukasiewicz logic. *Fuzzy Sets and Systems*, 159(10):1260–1267, 2008.