GEOMETRICAL DUALITIES FOR ŁUKASIEWICZ LOGIC

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Abstract. This article develops a general dual adjunction between MV-algebras (the algebraic equivalents of Łukasiewicz logic) and subspaces of Tychonoff cubes, endowed with the transformations that are definable in the language of MV-algebras. Such a dual adjunction restricts to a duality between semisimple MV-algebras and closed subspaces of Tychonoff cubes. Further the duality theorem for finitely presented objects is obtained from the general adjunction by a further specialisation. The treatment is aimed at emphasising the generality of the framework considered here in the prototypical case of MV-algebras.

Keywords: Łukasiewicz logic, MV-algebras, adjunction, categorical duality, Tychonoff cube, compact Hausdorff spaces, Hölder's theorem, Chang's completeness theorem, Wójcicki's theorem, rational polyhedra, piecewise linear maps, $\mathbb{Z}$-maps.
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1. Introduction.

This is the transcript of a *featured talk* given on the 15^{th} of September 2011 at the XIX Congeresso dell'Unione Matematica Italiana held in Bologna, Italy. It is based on a joint work with Vincenzo Marra of the University of Milan that was published in [11]. During the lecture proofs were given just as hints, so also this document does not carry any formal detail other than the complete statements of the results. When it is possible hints at the proof strategy as well as the concepts involved are given. For complete proofs of all unreferenced statements here, the reader should kindly refer to [11] and references therein.

This work was started with the hope of getting a better insight on relationship between finitely presented MV-algebras and rational polyhedra and how this can be framed in a more general perspective about geometric dualities. Indeed, in the last decade a number of results in the theory of MV-algebras were proved associating to each finitely presented structure an algebra of special functions (called McNaughton functions) restricted to a rational polyhedron contained in a hypercube $[0,1]^n$. It was already known to the practitioners of the field that this correspondence can be made functorial, but previous studies concentrated on its applications rather than on its own nature.

The outcomes of this study are:

(1) The relation between finitely presented MV-algebras and rational polyhedra is a categorical duality (see Theorem 8.5) which stems from a more general categorical duality between semisimple MV-algebras and Tychonoff spaces with *definable maps* (see Corollary 6.2).
(2) This duality arises exactly in the same way as Stone's dualities and can be seen as a generalisation of the adjunction between affine varieties over an algebraically closed field, and their structure rings (see Lemma 3.8).

(3) The general adjunction from which this correspondence arises seems to be a quite more general phenomenon that can be formally carried out in any given variety of algebras, once some special element in this variety has been fixed (see Theorem 3.11). Depending on the properties enjoyed by this distinguished element the general adjunction can be strengthen to categorical duality.

We will start form item (3). Although we treat the particular case of MV-algebras, it will transpire that this is a completely abstract construction which can be carried out in any variety. As mentioned in item (2) the construction is formally identical to the classical adjunction between affine algebraic varieties and polynomial ideals.

In his two landmark papers [14, 15], Marshall Stone showed that the set of prime (=maximal) ideals of a Boolean algebra carries a natural topology, one in which the open sets correspond to arbitrary ideals. Spaces arising in this manner are known today as *Stone spaces.* The *clopen* sets, i.e. closed and open in the topology, correspond to principal ideals, and hence to elements of the algebra. Thus, the original algebra can be recovered from its space of prime ideals; the bridge is in fact a two-way road. It is a simple exercise to rephrase also Stone's dualities in the framework of this general adjunction.

To get to item (1) we will gradually involve results coming from the general theory of MV-algebras. Although we will not refrain from using technical lemmas specifically valid for these structures, it will again transpire that a handful of basic properties of the distinguished algebra $[0, 1]$ are all is needed to prove a characterisation of the fixed points of this adjunction, whence the duality will follow as an immediate corollary.

We then turn to the full subcategory of finitely presented MV-algebras. Using further advanced results about MV-algebras, we obtain a duality between the category of finitely presented MV-algebras and the category of *finitely definable* subspaces of Tychonoff cubes. Finally, we give a characterisation in geometrical terms of the abstract category of finitely definable sets. This yields the geometric duality between finitely presented MV-algebras and the category of *rational polyhedra* with $\mathbb{Z}$*-maps* as morphisms. The result that affords this characterisation is *McNaughton's Theorem*, which allows us to identify definable maps on rational polyhedra with piecewise linear maps having integer coefficients.

2. Preliminaries.

MV-algebras are the equivalent algebraic semantics of *Łukasiewicz logic*, a many-valued propositional system going back to the 1920's. Chang [4] first singled out the axioms of *MV-algebras* by studying the structure of classes of equivalent formulas of Łukasiewicz logic (aka the Lindenbaum-Tarski algebra of the logic). Shortly thereafter, in his ground-breaking paper [5], he obtained an algebraic proof of the completeness theorem. The standard reference for the elementary theory of MV-algebras is [6], whereas [12] is a treatment at the frontier of current research.

Let us recall that an MV-algebra is an algebraic structure $(M, \oplus, \neg, 0)$, where $0 \in M$ is a constant, $\neg$ is a unary operation satisfying $\neg\neg x = x$, $\oplus$ is a unary operation making $(M, \oplus, 0)$ a commutative monoid, the element 1 defined as $\neg 0$ satisfies $x \oplus 1 = 1$, and the law

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \qquad (*)$$

holds. Any MV-algebra has an underlying structure of distributive lattice bounded below by 0 and above by 1. Joins are defined as $x \vee y = \neg(\neg x \oplus y) \oplus y$. Thus, the characteristic law (*) states that $x \vee y = y \vee x$. Meets are defined by the De Morgan condition $x \wedge y = \neg(\neg x \vee \neg y)$. Boolean algebras are precisely those MV-algebras that are idempotent, meaning that $x \oplus x = x$ holds, or equivalently, that satisfy the *tertium non datur* law $x \vee \neg x = 1$.

The interval (of truth values) $[0,1] \subseteq \mathbb{R}$ can be made into an MV-algebra, often called the *standard* MV-algebra. It has 0 as neutral element, $x \oplus y = \min\{x+y, 1\}$, and $\neg x = 1 - x$. The underlying lattice order of this MV-algebra coincides with the natural order that $[0,1]$ inherits from the real numbers.

A key point in Stone's duality for Boolean algebra is that prime ideals are the kernels of the homomorphisms into the prototypical Boolean algebra $\{0,1\}$. So Stone's duality asserts that the original structure of the Boolean algebra can be recovered from the information on the ways a Boolean algebra can be mapped homomorphically into $\{0,1\}$. In MV-algebras a similar role is played by the algebra [0,1]. The analogy is not complete in that all Boolean algebras are semisimple while there are non semisimple MV-algebras. This is the reason why the general adjunction for MV-algebras needs to be restricted to semisimple algebras to become a duality while it is already so in the framework of Boolean algebras.

We will be concerned here with the category $\mathsf{MV_p}$ of *presented MV-algebras*, i.e. the category whose objects are MV-algebras of the form $\mathcal{F}_\mu/\theta$, where μ is a cardinal, $\mathcal{F}_\mu$ is the MV-algebra freely generated by the set $\{X_\alpha \mid \alpha < \mu,\ \alpha \text{ an ordinal}\}$, and θ is a congruence on $\mathcal{F}_\mu$; morphisms are homomorphisms of MV-algebras. Using the Axiom of Choice, it is an exercise to show that $\mathsf{MV_p}$ is equivalent to the category of all MV-algebras. Therefore, our duality results extend to the category of abstract MV-algebras, too. It will transpire in the course of the development, however, that an extension obtained in this manner carries no genuine new mathematical information: we know of no way of associating to an abstract MV-algebra its dual object, as constructed in this paper, other than by arbitrarily choosing a presentation of the algebra. Thus, we opt for honesty and work with presented algebras throughout.

Notation. Throughout, μ and ν invariably denote cardinal numbers, whereas α and β invariably denote ordinal numbers. Although elements of $\mathcal{F}_\mu$ are equivalence classes of terms in the language of MV-algebras, we often use single terms as representatives for their equivalence classes. If s is a term, the notation $s\big((X_\alpha)_{\alpha<\mu}\big)$ means that the (finitely many) variables occurring in s are among those in the tuple $(X_\alpha)_{\alpha<\mu}$. If $s\big((X_\alpha)_{\alpha<\mu}\big) \in \mathcal{F}_\mu$ and $\{t_\alpha\}_{\alpha<\mu} \subseteq \mathcal{F}_\nu$, we denote by $s\big(\ [X_\alpha \backslash t_\alpha]_{\alpha<\mu}\ \big)$ the term obtained from s by uniformly replacing each variable X_α with the term t_α. Obviously, $s\big(\ [X_\alpha \backslash t_\alpha]_{\alpha<\mu}\ \big) \in \mathcal{F}_\nu$. We write $[0,1]^\mu$ for the Cartesian product of μ copies of $[0,1]$. If $p \in [0,1]^\mu$, then $s(p)$ denotes the evaluation of the term s in the MV-algebra $[0,1]$ under the assignment $X_\alpha \mapsto \pi_\alpha(p)$, where $\pi_\alpha \colon [0,1]^\kappa \to [0,1]$ is the projection onto the α^{th} coordinate, for each ordinal $\alpha < \kappa$.

We finally introduce the arrows of all our dual categories. The reader will readily see how this definition is just a specific instance of a classical concept considered in mathematical logic.

Definition 2.1. Given $S \subseteq [0,1]^\mu$ and $T \subseteq [0,1]^\nu$, a function $\lambda \colon S \to T$ is *definable* if there exists a ν-tuple of terms $(l_\beta)_{\beta<\nu}$, with $l_\beta \in \mathcal{F}_\mu$, such that

$$\lambda(\,(p_\alpha)_{\alpha<\mu}\,) = (\,l_\beta(\,(p_\alpha)_{\alpha<\mu}\,)\,)_{\beta<\nu}$$

for every $(p_\alpha)_{\alpha<\mu} \in S$. We call any such ν-tuple a family of *defining terms* for λ. In the special case that $\nu = 1$, the ν-tuple may be regarded as a single term $l \in \mathcal{F}_\mu$, called a *defining term* for λ.

We will denote by $\mathsf{T}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$ the category whose objects are subsets of $[0,1]^\mu$ as μ ranges over all cardinals and arrows are definable functions.

3. The basic adjunction.

Our first aim is to construct a pair of adjoint functors

$$\mathscr{I}\colon\ \mathsf{T}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}} \longrightarrow \mathsf{MV_p}\ , \quad \mathscr{V}\colon\ \mathsf{MV_p} \longrightarrow \mathsf{T}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}\ .$$

The construction of the functors as well the proof of the categorical adjunction is completely universal-algebraic: nothing more is needed beyond picking up an algebra in the variety, which in our case will be the distinguished algebra $[0,1]$ with its own MV-algebraic structure.

Definition 3.1 (The functor $\mathscr{I}$ on objects)**.** Given $S \subseteq [0,1]^\mu$, let us define a relation $\mathbb{I}\,(S)$ on $\mathcal{F}_\mu$ by stipulating that, for arbitrary terms $s,t \in \mathcal{F}_\mu$,

$$(s,t) \in \mathbb{I}\,(S) \text{ if and only if } [0,1] \models s(p) \approx t(p)$$

for every $p \in S \subseteq [0,1]^\mu$.

When $S = \{p\}$ is a singleton, we write $\mathbb{I}\,(p)$ in place of $\mathbb{I}\,(\{p\})$. For any $S \subseteq [0,1]^\mu$, it is easy to check that $\mathbb{I}\,(S)$ is a congruence on $\mathcal{F}_\mu$. In view of this, for any subset $S \subseteq [0,1]^\mu$ we define

$$\mathscr{I}\,(S) = \mathcal{F}_\mu\,/\,\mathbb{I}\,(S)\ .$$

Definition 3.2 (The functor $\mathscr{I}$ on arrows)**.** Given $S \subseteq [0,1]^\mu$ and $T \subseteq [0,1]^\nu$, let $\lambda\colon S \to T$ be a definable map, and let $(l_\beta)_{\beta<\nu}$ be a ν-tuple of defining terms for λ. Then there is an induced function

$$\mathscr{I}\,(\lambda)\colon\ \mathscr{I}\,(T) \to \mathscr{I}\,(S)$$

which acts on each $s \in \mathcal{F}_\nu$ by substitution as follows:

$$\frac{s\big(\,(X_\beta)_{\beta<\nu}\,\big)}{\mathbb{I}\,(T)} \in \mathscr{I}\,(T) \quad \overset{\mathscr{I}\,(\lambda)}{\longmapsto} \quad \frac{s\big(\,[X_\beta\backslash l_\beta]_{\beta<\nu}\,\big)}{\mathbb{I}\,(S)} \in \mathscr{I}\,(S)\ .$$

Remark 3.3. *1.* There can be several distinct defining terms for a definable function $\lambda\colon S \to [0,1]$. However, when l defines λ for each $p \in S$ one has $\lambda(p) = l(p)$, hence any pair of defining terms for the same definable function belongs to $\mathbb{I}\,(S)$.

2. Since terms commute with substitutions, it is also clear that the definition of $\mathscr{I}\,(\lambda)$ above does not depend on the choice of the representing term s, for if s' is another term such that $(s,s') \in \mathbb{I}\,(T)$, then $s\big(\,[X_\beta\backslash l_\beta]_{\beta<\nu}\,\big)$ is congruent to $s'\big(\,[X_\beta\backslash l_\beta]_{\beta<\nu}\,\big)$ modulo $\mathbb{I}\,(S)$

3. The two items above together imply that the definition of $\mathscr{I}\,(\lambda)$ does not depend on the choice of the family of defining terms $(l_\beta)_{\beta<\nu}$ either.

Proposition 3.4 (Functoriality of $\mathscr{I}$)**.** *Let $\lambda_1\colon S_1 \to S_2$ and $\lambda_2\colon S_2 \to S_3$ be definable maps, where each S_i is a subset of $[0,1]^{\mu_i}$, for some cardinal μ_i, $i = 1,2,3$. Then*

(1) $\mathscr{I}(\lambda_1)\colon \mathscr{I}(S_2) \to \mathscr{I}(S_1)$ *is a homomorphism of MV-algebras.*
(2) If λ_1 *is the identity map, then so is* $\mathscr{I}(\lambda_1)$*.*
(3) $\mathscr{I}(\lambda_2 \circ \lambda_1) = \mathscr{I}(\lambda_1) \circ \mathscr{I}(\lambda_2)$*.*

Therefore $\mathscr{I}$ *is a functor from* $\mathsf{T}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$ *into* $\mathsf{MV_p}$*.*

Proof. These are all consequences of simple universal algebraic arguments. □

Definition 3.5 (The functor $\mathscr{V}$ on objects)**.** Given $R = \{(s_i, t_i) \mid i \in I\} \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$, for I an index set, we define the set

$$\mathbb{V}(R) = \{p \in [0,1]^\mu \mid [0,1] \models s_i(p) \approx t_i(p) \text{ for each } i \in I\}\,.$$

By the very definition of $\mathbb{V}$, for any congruence θ on $\mathcal{F}_\mu$ we have $\mathbb{V}(\theta) \subseteq [0,1]^\mu$. We therefore set

$$\mathscr{V}(\mathcal{F}_\mu/\theta) = \mathbb{V}(\theta)\,.$$

Definition 3.6 (The functor $\mathscr{V}$ on arrows)**.** Let $h\colon \mathcal{F}_\mu/\theta_1 \to \mathcal{F}_\nu/\theta_2$ be a homomorphism of MV-algebras. For each $\alpha < \mu$, let π_α be the projection term on the α^{th} coordinate, and let π_α/θ_1 denote the equivalence class of π_α modulo θ_1. Fix, for each α, an arbitrary $f_\alpha \in h(\pi_\alpha/\theta_1)$. For any $(p_\beta)_{\beta<\nu} \in \mathbb{V}(\theta_2)$, set

$$\mathscr{V}(h)((p_\beta)_{\beta<\nu}) = \big(f_\alpha(\,(p_\beta)_{\beta<\nu}\,)\big)_{\alpha<\mu}\,.$$

Also in this case one observes that the definition of $\mathscr{V}(h)$ does not depend on the choices of the f_α's.

Proposition 3.7 (Functoriality of $\mathscr{V}$)**.** *Let* $h\colon \mathcal{F}_\mu/\theta_1 \to \mathcal{F}_\nu/\theta_2$ *and* $i\colon \mathcal{F}_\nu/\theta_2 \to \mathcal{F}_\xi/\theta_3$ *be homomorphisms of MV-algebras. Then*
(1) The function $\mathscr{V}(h)$ *is a definable map from* $\mathbb{V}(\theta_2)$ *to* $\mathbb{V}(\theta_1)$*.*
(2) If h *is the identity map, then so is* $\mathscr{V}(h)$*.*
(3) $\mathscr{V}(i \circ h) = \mathscr{V}(h) \circ \mathscr{V}(i)$*.*

Just as happens in algebraic geometry the pair $\mathbb{V}$ and $\mathbb{I}$, seen as functions between the powersets of $\mathcal{F}_\mu \times \mathcal{F}_\mu$ and $[0,1]^\mu$, form a *Galois connection.*

Lemma 3.8 (Basic Galois connection)**.** *For each* $S \subseteq [0,1]^\mu$ *and* $R \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$*,*

$$R \subseteq \mathbb{I}(S) \text{ if, and only if, } S \subseteq \mathbb{V}(R)\,.$$

Proof. Direct inspection of the definitions. □

As an immediate corollary valid for any Galois connection (see [8] for further references), we get

Corollary 3.9. *For* $\mathbb{V}$ *and* $\mathbb{I}$ *as above one has:*
(1) For any $S_1, S_2 \subseteq [0,1]^\mu$*,*
  a) $S_1 \subseteq \mathbb{V}(\mathbb{I}(S_1))$*,*
  b) $S_1 \subseteq S_2$ *implies* $\mathbb{I}(S_2) \subseteq \mathbb{I}(S_1)$*,*
  c) $\mathbb{I}(\mathbb{V}(\mathbb{I}(S_1))) = \mathbb{I}(S_1)$*, and*
  d) $\mathbb{I}$ *reverses arbitrary unions:*

$$\mathbb{I}\Big(\bigcup_{i\in I} S_i\Big) = \bigcap_{i\in I} \mathbb{I}(S_i)\ ,$$

where S_i *is a subset of* $[0,1]^\mu$*, and* I *is an arbitrary index set.*

(2) For any $R_1, R_2 \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$,
- *a) $R_1 \subseteq \mathbb{I}(\mathbb{V}(R_1))$,*
- *b) $R_1 \subseteq R_2$ implies $\mathbb{V}(R_2) \subseteq \mathbb{V}(R_1)$,*
- *c) $\mathbb{V}(\mathbb{I}(\mathbb{V}(R_1))) = \mathbb{V}(R_1)$, and*
- *d) $\mathbb{V}$ reverses arbitrary unions:*

$$\mathbb{V}\left(\bigcup_{i \in I} R_i\right) = \bigcap_{i \in I} \mathbb{V}(R_i) \ ,$$

where R_i is a subset of $\mathcal{F}_\mu \times \mathcal{F}_\mu$, and I is an arbitrary index set.

Remark 3.10. A function $C\colon 2^A \to 2^A$, where 2^A is the powerset of a set A, is a *closure operator* on A [3, I.5.1] if it is *extensive* ($X \subseteq C(X)$ for each $X \in 2^A$), *isotone* ($X \subseteq Y$ implies $C(X) \subseteq C(Y)$ for each $X, Y \in 2^A$), and *idempotent* ($C(C(X)) = C(X)$ for each $X \in 2^A$). The preceding lemma shows that the composition $\mathbb{V} \circ \mathbb{I}$ is a closure operator on $[0,1]^\mu$, and the composition $\mathbb{I} \circ \mathbb{V}$ is a closure operator on $\mathcal{F}_\mu \times \mathcal{F}_\mu$.

Throughout the paper, we write $\mathbb{1}_O$ to denote the identity arrow on the object O of a category C, and $\mathbb{1}_\mathsf{C}$ to denote the identity functor on C. Further, we write composition as juxtaposition whenever convenient, e.g. we write $\mathscr{V}\mathscr{I}$ in place of $\mathscr{V} \circ \mathscr{I}$.

Theorem 3.11 (The basic adjunction between MV-algebras and spaces)**.** *The functor $\mathscr{V}\colon \mathsf{MV_p} \longrightarrow \mathsf{T}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$ is left adjoint to the functor $\mathscr{I}\colon \mathsf{T}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}} \longrightarrow \mathsf{MV_p}$. In symbols, $\mathscr{V} \dashv \mathscr{I}$.*

Proof. To prove the statement we must exhibit two natural transformations

$$\eta\colon \mathbb{1}_{\mathsf{MV_p}} \to \mathscr{I}\mathscr{V} \qquad \text{and} \qquad \varepsilon\colon \mathscr{V}\mathscr{I} \to \mathbb{1}_{\mathsf{T}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}}$$

called *unit* and *co-unit*. In the first case we need to exhibit components $\eta_{\frac{\mathcal{F}_\mu}{\theta_1}}$ and $\eta_{\frac{\mathcal{F}_\nu}{\theta_2}}$ for any two objects $\mathcal{F}_\mu/\theta_1$ and $\mathcal{F}_\nu/\theta_2$ of $\mathsf{MV_p}$ such that the following diagram commutes.

$$\begin{array}{ccc}
\mathscr{I}\mathscr{V}\left(\frac{\mathcal{F}_\mu}{\theta_1}\right) & \xrightarrow{\mathscr{I}\mathscr{V}(h)} & \mathscr{I}\mathscr{V}\left(\frac{\mathcal{F}_\nu}{\theta_2}\right) \\
\uparrow {\scriptstyle \eta_{\frac{\mathcal{F}_\mu}{\theta_1}}} & & \uparrow {\scriptstyle \eta_{\frac{\mathcal{F}_\nu}{\theta_2}}} \\
\frac{\mathcal{F}_\mu}{\theta_1} & \xrightarrow{\quad h \quad} & \frac{\mathcal{F}_\nu}{\theta_2}
\end{array}$$

Note that $\mathscr{I}\mathscr{V}(\mathcal{F}_\mu/\theta_1) = \mathscr{I}(\mathbb{V}(\theta_1)) = \mathcal{F}_\mu/\mathbb{I}(\mathbb{V}(\theta_1))$. By *1c*) in Lemma 3.8, $\theta_1 \subseteq \mathbb{I}(\mathbb{V}(\theta_1))$, hence it is readily seen that the canonical homomorphism from $\mathcal{F}_\mu/\theta_1$ to $\mathscr{I}\mathscr{V}(\mathcal{F}_\mu/\theta_1)$, which sends a generic element s/θ_1 of $\mathcal{F}_\mu/\theta_1$ into $s/\mathbb{I}(\mathbb{V}(\theta_1))$ suits the task. The arrow $\eta_{\mathcal{F}_\nu/\theta_2}$ is defined in the same way.

In the second case we need to show that for any two objects S and T of $\mathsf{T}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$ there are ε_S and ε_T such the following diagram commutes.

$$\begin{array}{ccc} \mathscr{V}\mathscr{I}(S) & \xrightarrow{\mathscr{V}\mathscr{I}(\lambda)} & \mathscr{V}\mathscr{I}(T) \\ \downarrow{\scriptstyle \varepsilon_S} & & \downarrow{\scriptstyle \varepsilon_T} \\ S & \xrightarrow{\lambda} & T \end{array}$$

Recall that $\mathscr{V}\mathscr{I}(S) = \mathscr{V}(\mathcal{F}_\mu \,/\, \mathbb{I}(S)) = \mathbb{V}(\mathbb{I}(S))$, so by *1a)* in Lemma 3.8 we can take the dual of the inclusion arrow $S \hookrightarrow \mathbb{V}(\mathbb{I}(S))$. The arrow ε_T is defined analogously.

Next, we need to show that for any $A \in \mathsf{MV_p}$ the diagram below commutes.

$$\mathscr{V}(A) \xrightarrow{\mathscr{V}(\eta_A)} \mathscr{V}\mathscr{I}\mathscr{V}(A) \xrightarrow{\varepsilon_{\mathscr{V}(A)}} \mathscr{V}(A), \qquad \mathbb{1}_{\mathscr{V}(A)} : \mathscr{V}(A) \to \mathscr{V}(A)$$

If $A = \mathcal{F}_\mu/\theta$, then $\mathscr{V}(\mathcal{F}_\mu/\theta) = \mathbb{V}(\theta)$ and $\mathscr{V}\mathscr{I}\mathscr{V}(\mathcal{F}_\mu/\theta) = \mathscr{V}\mathscr{I}(\mathbb{V}(\theta)) = \mathscr{V}(\mathcal{F}_\mu \,/\, \mathbb{I}(\mathbb{V}(\theta))) = \mathbb{V}(\mathbb{I}(\mathbb{V}(\theta)))$. Again the claim is settled by Lemma 3.8 which asserts that $\mathscr{V}(\mathcal{F}_\mu/\theta) = \mathscr{V}\mathscr{I}\mathscr{V}(\mathcal{F}_\mu/\theta)$, the rest is a matter of checking the definitions of the arrows.

Finally, for any $K \in \mathsf{T}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$, the diagram below commutes.

$$\mathscr{I}(K) \xrightarrow{\mathscr{I}(\varepsilon_K)} \mathscr{I}\mathscr{V}\mathscr{I}(K) \xrightarrow{\eta_{\mathscr{I}(K)}} \mathscr{I}(K), \qquad \mathbb{1}_{\mathscr{I}(K)} : \mathscr{I}(K) \to \mathscr{I}(K)$$

Again we have that $\mathscr{I}(K) = \mathbb{I}(\theta)$ and $\mathscr{I}\mathscr{V}\mathscr{I}(K) = \mathbb{I}(\mathbb{V}(\mathbb{I}(\theta)))$, which are equal by *2a)* in Lemma 3.8. □

4. The co-Nullstellensatz.

Once an adjunction has been established it is natural to look for the fixed points of the adjunction, i.e. objects on which the unit and co-unit of the adjunction are isomorphisms. The reason is that the adjunction restricted to the above objects is a *categorical duality*, hence an explicit description of the fixed points affords a complete characterisation of two subcategories which are equivalent. In categorical terms this translates in an explicit description of the co-unit and unit of the adjunction in Theorem 3.11, which is our aim in this section.

We will see that the analogue of the Zariski topology on affine space now enters the picture. If we let the definable subsets (i.e. the sets $V(\theta)$ for some congruence θ) be the closed sets of a topology, the result is precisely the Tychonoff (product) topology on $[0,1]^\mu$. This is the content of Lemma 4.3; we call it co-Nullstellensatz in that it is the category-theoretic dual of the analogue of Hilbert's Nullstellensatz in algebraic geometry.

The analogy with classical algebraic geometry is not merely a category-theoretic formality. Several results in algebraic geometry compare the Zariski (definable) topology on an affine variety (over $\mathbb{C}$) with the complex-analytic topology the variety inherits from its embedding into $\mathbb{C}^n$; for irreducible varieties, for example, it is well known that every Zariski-open set is a dense open set in the complex-analytic topology. Lemma 4.3 is precisely a result of this sort: it compares the notion of algebraically definable closed set with the "natural" notion of closed set in a Tychonoff cube, to find out that they coincide.

Here and henceforth we follow Engelking's treatise on topology [7]. From now on, each Cartesian product $[0,1]^\mu$ will be endowed with its *Tychonoff*, or *product*, topology, where on $[0,1]$ we assume the usual Euclidean topology. Recall that a topological space X is *completely regular*, or *Tychonoff* (or $T_{3\frac{1}{2}}$) if it is T_1, and points and closed sets can be separated by continuous $[0,1]$-valued functions: if $x \in X$, $Y \subseteq X$ is closed, and $x \notin Y$, there is a continuous function $f\colon X \to [0,1]$ such that $Y \subseteq f^{-1}(0)$, and $f(x) > 0$. It is a standard fact that $[0,1]^\mu$ is a Tychonoff space. We are going to prove in Lemma 4.2 below the stronger result that points and closed sets in $[0,1]^\mu$ can be separated by *definable* functions. As a final piece of notation, let us write $\overline{S}$, for $S \subseteq [0,1]^\mu$, to denote the closure of S in $[0,1]^\mu$, i.e. the intersection of all closed subsets of $[0,1]^\mu$ containing S.

Remark 4.1. Since the operations of the standard MV-algebra $[0,1]$ are continuos with respect to the Euclidean topology, it is clear that *any definable map* $\lambda\colon S \to T$ *is continuous*, where S and T are endowed with the subspace topology they inherit from the Tychonoff topology of $[0,1]^\mu$ and $[0,1]^\nu$, respectively. Thus, if we regard each object of $\mathsf{T}_{\mathsf{def}\,\mathbb{Z}}$ as a topological space with the subspace topology from $[0,1]^\mu$, $\mathsf{T}_{\mathsf{def}\,\mathbb{Z}}$ is a subcategory of the category of Tychonoff spaces and continuous maps.

Lemma 4.2 (Complete regularity by definable functions)**.** *For any point* $p \in [0,1]^\mu$ *and any closed set* $K \subseteq [0,1]^\mu$ *with* $p \notin K$*, there is a definable function* $\lambda\colon [0,1]^\mu \to [0,1]$ *that takes value* 0 *over* K*, and value* > 0 *at* p*.*

Proof. It follows immediately from [1, Corollary 2.8], that for each open interval $(a,b) \subseteq [0,1]$ and each $p \in (a,b)$, there are terms s and t such that the function $\lambda\colon [0,1] \to [0,1]$ defined by the term $s \wedge \neg t$ satisfies $\lambda(p) > 0$ and $(\,[0,1] \setminus (a,b)\,) \subseteq \lambda^{-1}(0)$. The statement then follows from this and the regularity of $[0,1]^\mu$. □

Lemma 4.3 (Co-Nullstellensatz for MV-algebras)**.** *For any* $S \subseteq [0,1]^\mu$*,*

$$\mathbb{V}(\mathbb{I}\,(S)) = \overline{S}\ .$$

So the set $S \subseteq [0,1]^\mu$ *is closed if, and only if,* $\mathbb{V}(\mathbb{I}\,(S)) = S$*.*

Proof. The inclusion $\overline{S} \subseteq \mathbb{V}\,(\mathbb{I}\,(S))$ follows from the fact that $S \subseteq \mathbb{V}\,(\mathbb{I}\,(S))$ always holds by Lemma 3.8, the fact that $[0,1]$ is Hausdorff and definable functions are continuous, hence the solution set of $f = g$ is a closed set.

The converse inclusion is an immediate consequence of Lemma 4.2 □

Remark 4.4. The co-Nullstellensatz implies in particular that $\mathbb{V} \circ \mathbb{I}$ is a topological closure operator, i.e. a closure operator satisfying $C(\bigcup_{i \in I} X_i) = \bigcup_{i\in I} C(X_i)$, because topological closure is.

5. The Nullstellensatz.

As next step we are going to characterise the action of the components of the co-unit η on presented MV-algebras. In this case the parallel with Hilbert's Nullstellesatz will be plain.

Let us recall some terminology from universal algebra applied to our case (for details see [3]). An MV-algebra is *simple* if it has no proper congruence but the identity congruence, and it is *semisimple* if it is a subdirect product of simple MV-algebras. Equivalently, an MV-algebra A is semisimple if the intersection of its maximal congruences, called *radical congruence* and denoted by $\mathrm{Rad}\,(A)$, is the identity relation.

We will show that the fixed points in this case are the algebras presented by a *radical* congruence. A crucial ingredient in the proof will be the fact that every simple algebra can be embedded in our distinguished algebra $[0,1]$. This is a folklore result which goes under the name of Hölder's Theorem for MV-algebras.

Lemma 5.1 (Hölder's Theorem for MV-algebras). *Let C be a non-trivial, simple MV-algebra. Then there is a unique injective homomorphism $C \to [0,1]$.*

Lemma 5.2 (Point=Maximal congruence). *For any set $S \subseteq [0,1]^\mu$, and for any congruence θ on $\mathcal{F}_\mu$, the following hold.*

(1) If θ is a maximal congruence then $\mathbb{V}\,(\theta)$ is a singleton.
(2) If S is a singleton then $\mathbb{I}\,(S)$ is a maximal congruence.

Proof. Both items can be proved using (the topological!) Lemma 4.2 and the fact that $\mathbb{I}$ and $\mathbb{V}$ are a Galois adjunction (Lemma 3.8). Lemma 5.1 is crucial in the proof of item *(1)*. □

Further, given $S \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$, let us write $\langle S\rangle$ for the congruence on $\mathcal{F}_\mu$ generated by S, i.e. the intersection of all congruences containing S.

Lemma 5.3 (Nullstellensatz for MV-algebras). *For any $S \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$,*

$$\mathbb{I}\,(\mathbb{V}\,(S)) = \bigcap\{\theta' \in (\mathcal{F}_\mu)^2 \mid S \subseteq \theta' \text{ and } \theta' \text{ maximal congruence}\} = \mathrm{Rad}\,(\mathcal{F}_\mu\,/\langle S\rangle)\ .$$

So the algebra $\mathcal{F}_\mu\,/\theta$ is semisimple if, and only if, $\mathbb{I}\,(\mathbb{V}\,(\theta)) = \theta$.

Proof. This is proved by combining Lemma 3.8 and Lemma 5.2. □

6. The categorical duality for semisimple MV-algebras.

The Nullstellensatz and co-Nullstellensatz above give an explicit characterisation of the fixed point of the categorical adjunction $\mathscr{I} \vdash \mathscr{V}$, we recapitulate them in the following.

Theorem 6.1 (Co-unit & Unit as Closure & Radical).

(1) The co-unit $\varepsilon\colon \mathscr{V}\,\mathscr{I} \to \mathbb{1}_{\mathsf{T}^{\mathrm{op}}_{\mathrm{def}\,\mathbb{Z}}}$ acts as the closure operator associated to the Tychonoff topology of $[0,1]^\mu$.
In other words, if $S \subseteq [0,1]^\mu$, the component $\varepsilon_S\colon \mathscr{V}\,\mathscr{I}\,(S) \to S$ is (the dual of) the inclusion arrow $S \hookrightarrow \overline{S}$ that embeds S in its closure.

(2) The unit $\eta\colon \mathbb{1}_{\mathsf{MV_p}} \to \mathscr{I}\,\mathscr{V}$ acts by modding out radicals.
In other words, if θ is a congruence on $\mathcal{F}_\mu$, the component $\eta_{\mathcal{F}_\mu/\theta}\colon \mathcal{F}_\mu\,/\theta \to \mathscr{I}\,\mathscr{V}\,(\mathcal{F}_\mu\,/\theta)$ is the natural quotient map $\mathcal{F}_\mu\,/\theta \twoheadrightarrow (\mathcal{F}_\mu\,/\theta)/\,\mathrm{Rad}\,(\mathcal{F}_\mu\,/\theta)$ induced by the congruence $\mathrm{Rad}\,(\mathcal{F}_\mu\,/\theta)$.

Hence,

(1) ε_S *is an isomorphism if, and only if, S is closed.*
(2) $\eta_{\mathcal{F}_\mu/\theta}$ *is an isomorphism if, and only if,* $\mathcal{F}_\mu/\theta$ *is semisimple.*

Let $\mathsf{MV}_\mathsf{p}^{\mathsf{ss}}$ be the full subcategory of MV_p whose objects are (presented) semisimple MV-algebras. Further, let $\mathsf{K}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$ be the full subcategory of $\mathsf{T}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$ whose objects are closed subsets of Tychonoff cubes.

As mentioned at the beginning of this section Theorem 6.1 entails at once a categorical duality:

Corollary 6.2 (Duality theorem for semisimple MV-algebras)**.** *The adjunction* $\mathscr{V} \dashv \mathscr{I}$ *in Theorem 3.11 restricts to an equivalence of categories between* $\mathsf{MV}_\mathsf{p}^{\mathsf{ss}}$ *and* $\mathsf{K}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$.

Let us conclude with an important remark. Each space in $\mathsf{K}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$ is compact and Hausdorff [7, 3.1.2, 2.1.6]; conversely, each compact Hausdorff space can be embedded in some Tychonoff cube [7, 2.3.23]. It should be carefully noted, however, that the notion of definable map between compact Hausdorff spaces H and K only makes sense if H and K come endowed with a *specific* embedding into $[0,1]^\mu$ and $[0,1]^\nu$, respectively. In other words, an object of $\mathsf{K}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$ cannot be conceived of as an abstract compact Hausdorff space K, but is rather a continuous embedding $K \hookrightarrow [0,1]^\mu$.

7. The duality for finitely presented MV-algebras.

For the rest of this paper, we let m be a non-negative integer. The aim of this section is show that the adjunction given by the pair $\mathscr{I}, \mathscr{V}$ restricts to an equivalence for finitely presented algebras. To this end we introduce the full subcategory of $\mathsf{T}_{\mathsf{def}\,\mathbb{Z}}$ which is the $\mathscr{V}$-image of finitely presented algebras.

Definition 7.1. A subset $S \subseteq [0,1]^\mu$ is called *finitely definable* if there is a finite index set I, along with a set of pairs $R = \{(s_i, t_i) \in \mathcal{F}_\mu \times \mathcal{F}_\mu \mid i \in I\}$, such that $S = \mathbb{V}(R)$. The full subcategory of $\mathsf{T}_{\mathsf{def}\,\mathbb{Z}}$ whose objects are finitely definable subsets of $[0,1]^m$, as m ranges over all natural numbers, is denoted $\mathsf{D}_{\mathsf{def}\,\mathbb{Z}}$.

Remark 7.2. In MV-algebras, finitely generated and principal (=singly generated) congruences coincide.

Next Lemma shows that finitely definable sets coincide with the vanishing loci of compact congruences. The proof requires two related non-trivial results from the theory of MV-algebras: a geometrical characterisation of principally generated ideals, asserting that if s, t, u, v are elements of $\mathcal{F}_m$, then

$$(u,v) \in \langle (s,t) \rangle \text{ if, and only if, } \mathbb{V}\,(s,t) \subseteq \mathbb{V}(u,v)\ ,$$

and Wójcicki's Theorem

Theorem 7.3 (Wójcicki's Theorem)**.** *Every finitely presented MV-algebra is semisimple.*

Putting these two results together we have

Lemma 7.4 (Finitely definable set=Compact congruence)**.**

(1) *If* $D \subseteq [0,1]^m$ *is a finitely definable set, then* $\mathbb{I}\,(D) \subseteq \mathcal{F}_m \times \mathcal{F}_m$ *is a finitely generated congruence.*

(2) If $\theta \subseteq \mathcal{F}_m \times \mathcal{F}_m$ is a finitely generated congruence, then $\mathbb{V}(\theta) \subseteq [0,1]^m$ is a finitely definable set.

As in the previous section, this immediately entails the following.

Theorem 7.5. *The adjunction $\mathscr{V} \dashv \mathscr{I}$ restricts to an equivalence of categories between $\mathsf{MV}_{\mathsf{fp}}$ and $\mathsf{D}^{\mathrm{op}}_{\mathsf{def}\,\mathbb{Z}}$.*

8. Concrete description of the categorical dual of finitely presented MV-algebras.

The abstract category $\mathsf{D}_{\mathsf{def}\,\mathbb{Z}}$ can be characterised in purely geometrical terms. This yields the geometric duality between finitely presented MV-algebras and rational polyhedra. For the general background on polyhedra, see [13].

If $S \subseteq \mathbb{R}^m$ is any subset, we let $\mathtt{conv}\, S$ denote the *convex hull* of S. Recall that a *polytope* is a subset of $\mathbb{R}^m$ of the form $\mathtt{conv}\, S$, for some finite $S \subseteq \mathbb{R}^m$, and a (*compact*) *polyhedron* is a union of finitely many polytopes in $\mathbb{R}^m$. A polytope $\mathtt{conv}\, S$ is *rational* if $S \subseteq \mathbb{Q}^m$. Similarly, a polyhedron is *rational* if it may be written as a union of finitely many rational polytopes.

Definition 8.1. We call Z-map a continuous *piecewise linear function with integer coefficients* from a rational polyhedron $P \subseteq [0,1]^m$ into $[0,1]^n$.

We denote the category of rational polyhedra and Z-maps by $\mathsf{P}_{\mathbb{Z}}$. The key fact is that $\mathbb{Z}$-maps between rational polyhedra are precisely the definable maps.

Lemma 8.2 (McNaughton's Theorem for rational polyhedra)**.** *Let $P \subseteq [0,1]^m$ be a rational polyhedron, and let $\lambda\colon P \to [0,1]$ be any function. Then λ is a $\mathbb{Z}$-map if, and only if, λ is a definable function.*

Remark 8.3. Lemma 8.1 may well fail for more general sets than rational polyhedra. Indeed, while definable maps are always $\mathbb{Z}$-maps, the converse inclusion does not hold in general.

Lemma 8.4. *The category $\mathsf{D}_{\mathsf{def}\,\mathbb{Z}}$ coincides with the category $\mathsf{P}_{\mathbb{Z}}$.*

Proof. Note preliminarily the general fact that S is a rational polyhedron if, and only if, there is a $\mathbb{Z}$-map $\zeta\colon [0,1]^m \to [0,1]$ vanishing precisely on S. By Remark 7.2, one has that any finitely definable set S is the solution set over $[0,1]^m$ of the equation $s = 0$. By Lemma 8.2, s is a $\mathbb{Z}$-map, and therefore S is a rational polyhedron. Conversely, if S is a rational polyhedron in $[0,1]^m$, there is a $\mathbb{Z}$-map $\zeta\colon [0,1]^m \to [0,1]$ such that $\zeta^{-1}(0) = S$. By Lemma 8.2 there is a term $s \in \mathcal{F}_m$ such that ζ is the function defined by s, and therefore, since $S = \mathbb{V}(s, 0)$, S is finitely definable. □

Corollary 8.5 (The duality theorem for finitely presented MV-algebras[1])**.** *The adjunction $\mathscr{V} \dashv \mathscr{I}$ in Theorem 3.11 restricts to an equivalence of categories between $\mathsf{MV}_{\mathsf{fp}}$ and $\mathsf{P}^{\mathrm{op}}_{\mathbb{Z}}$.*

Proof. Immediate consequence of Theorem 7.5 and Lemma 8.4. □

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[1] This result was already known to practitioners of the field, although we know of no detailed argument given in literature. An alternative proof was recently given in [10], although this aims at succinctness rather than investigating the origin of the correspondence.

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