Dualities and Geometry

General affine adjunctions, Nullstellensätze, and transforms

Joint work with O. Caramello and V. Marra

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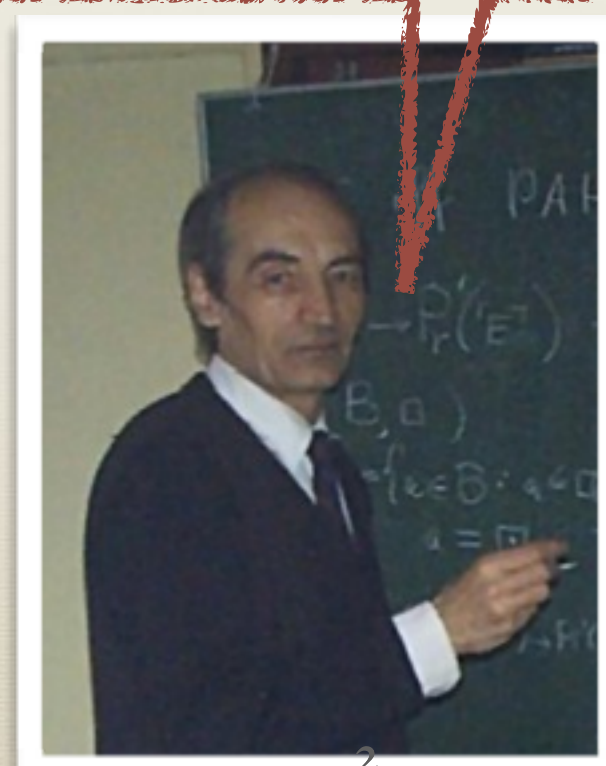
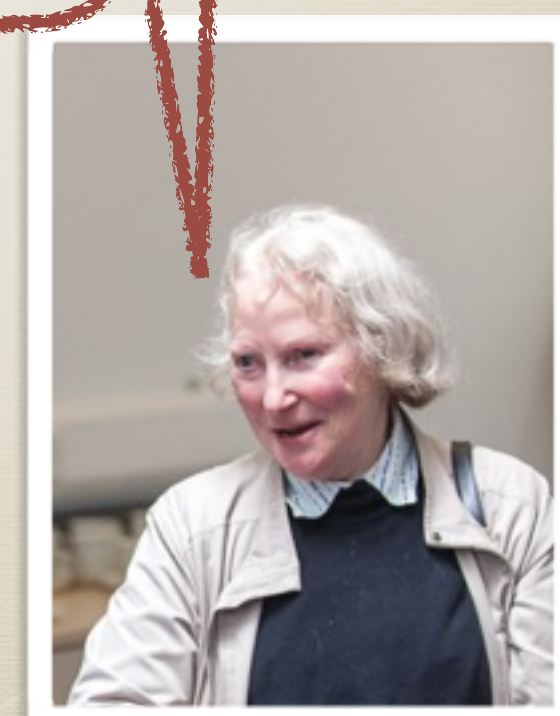
Dualities in Logic…

Boolean algebras and compact Hausdorff zero-dimensional spaces are dually equivalent.

Distributive lattices are dually equivalent to totally order disconnected compact Hausdorff spaces.

Heyting algebras are dually equivalent to Esakia spaces.



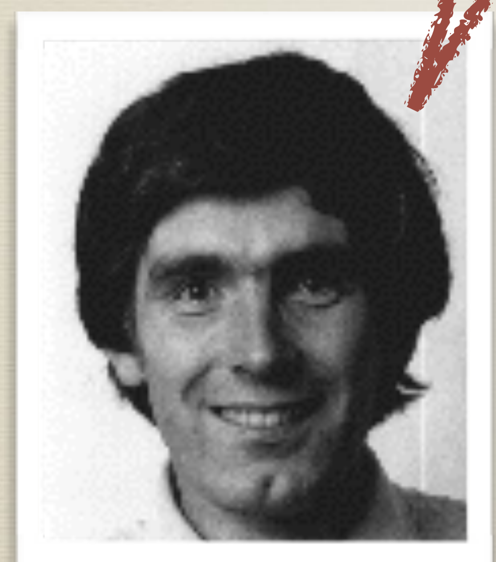


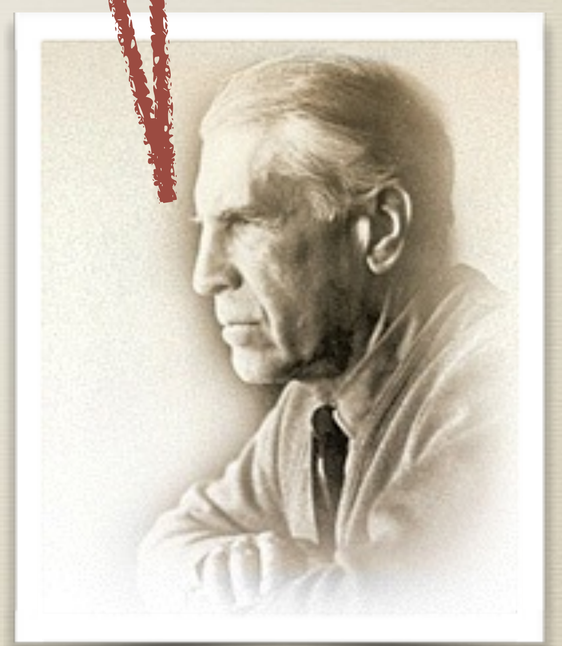
… in Mathematics…

F.p. Riesz spaces are dually equivalent to polyhedral fans.

Compact abelian groups are dually equivalent to abelian groups.

Commutative unital C*-algebras are dually equivalent to compact Hausdorff spaces.



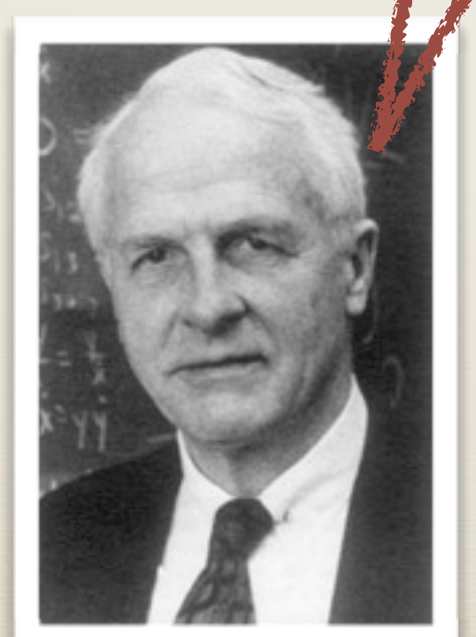


Algebra, Geometry and Functional Analysis

$$\mathbb{C}(\mathbb{V}(I)) = \sqrt{I}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) e^{ixt}\, dt$$

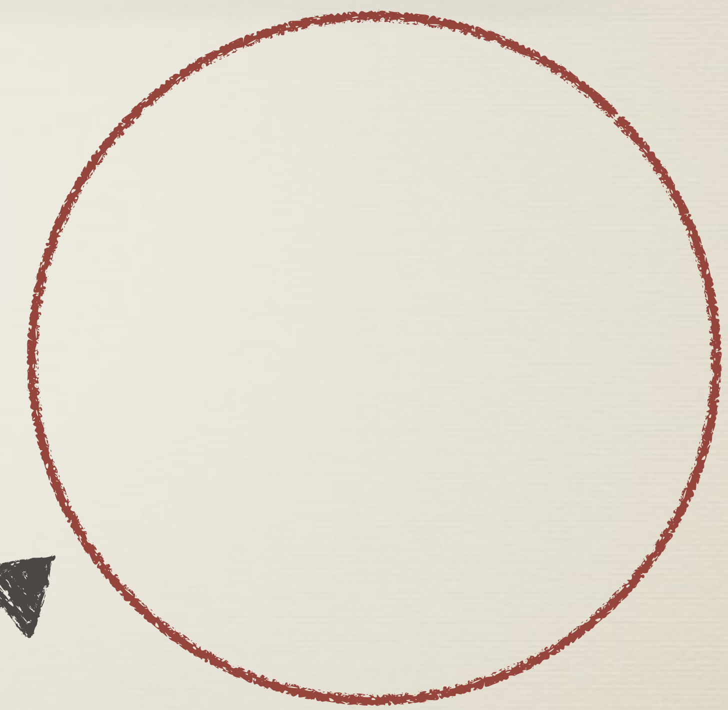
Every algebra in a finitary variety is decomposable in subdirectly irreducible algebras.



The fundamental adjunction in algebraic geometry

Polynomials that are zero on these points

$$x^2 + y^2 - 1$$
$$2x^2 + 2y^2 - 2$$
$$(x^2 + y^2 - 1) + (x^2 + y^2 - 1)$$



Points on which these polynomials are zero

The fundamental adjunction in algebraic geometry

1. k algebraically closed field
2. $k[x_1, ..., x_n]$ ring of polynomials in *n* variables

$$\mathbb{C}(S) = \{p \in k[x_1, ..., x_n] \mid \forall s \in S \quad p(s) = 0\}$$

$$\wp(k[x_1, ..., x_n]) \qquad \wp(k^n)$$

$$\mathbb{V}(\theta) = \{s \in k^n \mid \forall p \in \theta \quad p(s) = 0\}$$

The fundamental adjunction in algebraic geometry

1. Any algebra A in the variety $\mathcal{V}$
2. The free algebra $\mathcal{F}(\mu)$ in $\mathcal{V}$

$$\mathbb{C}(S) = \{p \in k[x_1, ..., x_n] \mid \forall s \in S \quad p(s) = 0\}$$

$$\wp\big((\mathcal{F}(\mu))^2\big) \qquad \wp\big(A^\mu\big)$$

$$\mathbb{V}(\theta) = \{s \in k^n \mid \forall p \in \theta \quad p(s) = 0\}$$

Affine adjunction in general algebra

1. Any algebra A in the variety $\mathcal{V}$
2. The free algebra $\mathcal{F}(\mu)$ in $\mathcal{V}$

$$\mathbb{C}(S) = \{(p,q) \in \mathcal{F}(\mu) \mid \forall s \in S \quad p(s) = q(s)\}$$

$$\wp\big((\mathcal{F}(\mu))^2\big) \qquad \wp\big(A^\mu\big)$$

$$\mathbb{V}(\theta) = \{s \in A^\mu \mid \forall (p,q) \in \theta \quad p(s) = q(s)\}$$

Affine adjunction in general algebra

Lemma. For any variety V and for any algebra A in V, the operators $\mathbb{V}$ and $\mathbb{C}$ form a Galois connection i.e.,

$$\forall S \subseteq A^{\mu} \quad \forall \theta \subseteq (\mathcal{F}(\mu))^2$$

$$\boxed{S \subseteq \mathbb{V}(\theta) \quad \Longleftrightarrow \quad \theta \subseteq \mathbb{C}(S)}$$

A reference table

Algebraic geometry	General algebra
Ground field k	A
Ring of polynomials k[x	$\mathcal{F}(\mu)$
Affine space k	A^{μ}
Ideals of k[x	Congruences of $\mathcal{F}(\mu)$
Coordinate rings k[x $\mathbb{C}(\mathbb{V}(I))$	Quotients $\mathcal{F}(\mu)/\mathbb{C}(\mathbb{V}(\theta))$.
Homomorphisms of k-algebras	$\mathcal{V}$-homomorphisms
Affine maps	**????**

Definable functions

Definition. A function $f\colon A^\mu \to A$ is called definable if there exist a term $t(\mathbf{x})$ such that

$$\forall \mathbf{a} \in A^\mu \qquad f(\mathbf{a}) = t(\mathbf{a})$$

More generally, a function $f\colon S \subseteq A^\mu \to T \subseteq A^\nu$ is called definable if there exist a family of terms $\{t_\beta(\mathbf{x})\}_{\beta<\nu}$ such that

$$\forall \mathbf{a} \in S \quad f(\mathbf{a}) = \big(t_\beta(\mathbf{a})\big)_{\beta<\nu}$$

Two categories

Definition. The category V is given by algebras in $\mathcal{V}$ of the form $\mathcal{F}(\mu)/\theta$, for μ ranging among cardinals, and homomorphisms among them.

Definition. The category D is by subsets of A^{μ}, for μ ranging among cardinals, and definable maps among them.

Affine adjunction

Theorem. The operators $\mathbb{C}$ and $\mathbb{V}$ induce a dual adjunction between the categories V and D.

Affine adjunction

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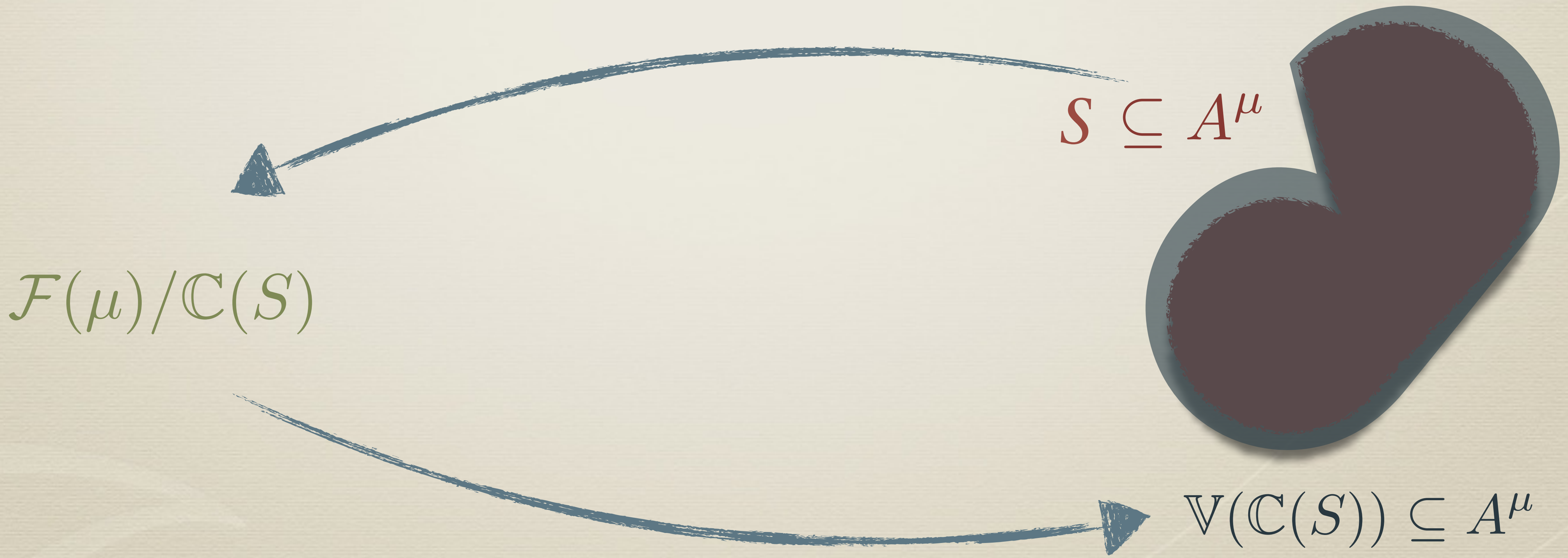
A dual equivalence

Remark. Just as any Galois connection subsumes an isomorphism of posets between its fixed points, any categorical adjunction subsumes an equivalence among its fixed points.



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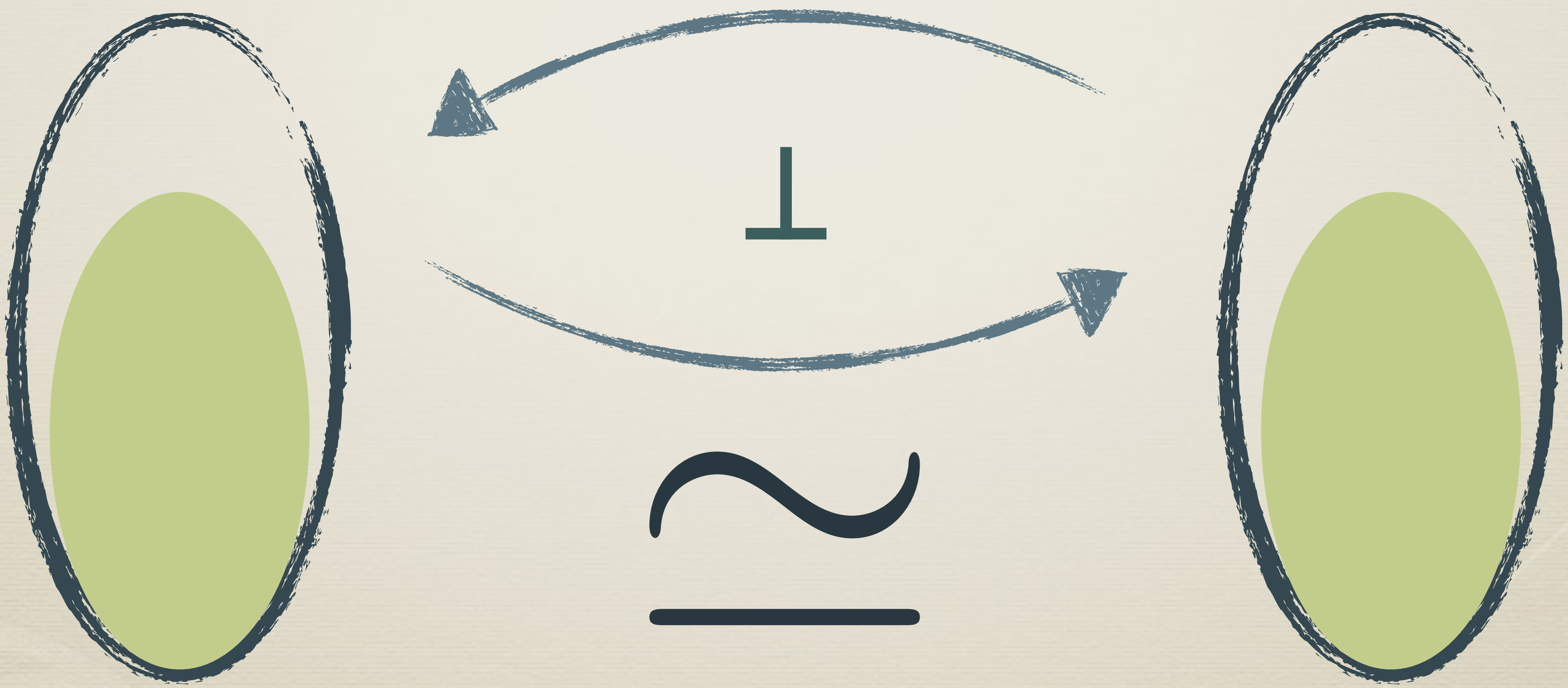


The fixed points: algebraic side

Theorem [*Algebraic Nullstellensatz*]**.** For any variety V, for any algebra A in V, and for any subset $\theta \subseteq (\mathcal{F}(\mu))^2$ the following are equivalent:

A. The subset θ is fixed i.e., $\mathbb{C}(\mathbb{V}(\theta)) = \theta$

B. $\theta = \bigcap_{\mathbf{a} \in \mathbb{V}(\theta)} \mathbb{C}(\mathbf{a})$

C. The map $\beta\colon \mathcal{F}(\mu)/\theta \to \prod_{\mathbf{a} \in \mathbb{V}(\theta)} \mathcal{F}(\mu)/\mathbb{C}(\{\mathbf{a}\})$ is a subdirect embedding.

The fixed points: algebraic side

Theorem [*GKS-Lemma*]**.** For any variety V, for any algebra A in V, and for any congruence $\theta \subseteq (\mathcal{F}(\mu))^2$ the following are equivalent:

A. There exists $\mathbf{a} \in A^{\mu}$ such that $\theta = \mathbb{C}(\{\mathbf{a}\})$.

B. The *Gel'fand evaluation* $\gamma_a : \mathcal{F}(\mu)/\theta \to A$ is an embedding.

The fixed points: geometric side

* The fixed points on the geometric side are sets of the form $\mathbb{V}(\mathbb{C}(S))$.
* Notice that such a composition is a closure operator.
* In several cases (all the ones mentioned here) it is a *topological* closure operator i.e., the sets $\mathbb{V}(\mathbb{C}(S))$ correspond to closed sets in some topology —in fact the *Zariski* topology.

characterise fixed points

=

characterise closed spaces in the Zariski topology of every power of A.

The fixed points: geometric side

Let X, $\{Y_\alpha \mid \alpha \in \mathfrak{A}\}$ be non empty topological spaces. Let $\mathcal{F}$ be a family of maps f_α each one from X into Y_α.
Define

$$\varepsilon\colon X \to \prod_{\alpha\in\mathfrak{A}} Y_\alpha \quad \text{by} \quad \varepsilon(p) := \big(f_\alpha(p)\big)_{\alpha\in\mathfrak{A}}$$

Theorem [Kelley's Embedding Lemma (1955)]. The evaluation map is:

1. *Continuous* **iff** all functions in $\mathcal{F}$ are continuous.
2. *Injective* **iff** $\mathcal{F}$ separates points.
3. *Open* in $\varepsilon[X]$ **if** $\mathcal{F}$ separates closed sets and points.

Case study: Boolean algebras

* Fix $\mathcal{V}$= Boolean algebras and A=$\{0,1\}$
* The Boolean algebra $\{0,1\}$ is the only subdirectly irreducible algebra in $\mathcal{V}$.
* Use GKS-lemma to deduce that every congruence that presents a subdirectly irreducible algebra is of the form $\mathbb{C}(\{a\})$.
* Use general algebra to argue that any congruence is intersection of congruences that present subdirectly irreducible algebras.
* Use the algebraic Nullstellensatz to derive that *every Boolean congruence is fixed.*

… and their dual spaces

* The category D is given by subsets of $\{0,1\}^\mu$ with definable maps.
* The sets $\{0,1\}^\mu$ get topologised with Zariski topology.
* Simple calculations show that *all* closed spaces in this category are Stone spaces and definable maps are *all* continuous maps among them.
* Use Kelley's Lemma to prove that *every* Stone space embeds into some closed subspace of these cubes.
* All together this shows that the category D is equivalent to the one of Stone spaces with continuous maps.

Stone duality for Boolean algebras

Category of Boolean algebras with their homomorphisms

$\simeq$

Category of *presented* Boolean algebras with their homomorphisms

$\simeq^{\mathrm{op}}$

Category of closed subsets of $\{0,1\}^{\mu}$ with definable maps

$\simeq$

Category of Stone spaces with continuous maps

Easy exercise (requires AC)

The duality $\mathbb{V}, \mathbb{C}$

Shown in last slide

A further generalisation

Fix the following data:

* A functor between locally small categories $\mathcal{I} \colon \mathtt{T} \to \mathtt{S}$
* An object $\triangle$ in $\mathtt{T}$

Define the following two categories:

* R has
  - objects: pairs (t,R) with t in T and $R \subseteq \hom^2(t, \triangle)$.
  - arrows: an arrow from (t,R) to (t',R') is an arrow in T from t to t' that is *compatible* with R and R'.
* D has
  - objects: pairs (t,s) with s subobject of $\mathcal{I}(t)$.
  - arrows: an arrow from (t,s) to (t',s') is an arrow in T from t to t' that is *compatible* with s and s'.

A further generalisation

$$\mathbb{C}(s) := \left\{(p, q) \in \hom^2_{\mathbb{T}}(t, \triangle) \mid \mathcal{I}(p) \circ s = \mathcal{I}(q) \circ s\right\}$$

$$\mathbb{V}(R) := \bigwedge_{(p,q)\in R} \mathrm{Eq}(\mathcal{I}(p), \mathcal{I}(q)),$$

A further generalisation

Theorem [*Abstract Nullstellensatz*].
Suppose S=Set. For any object (t,R) the following are equivalent.

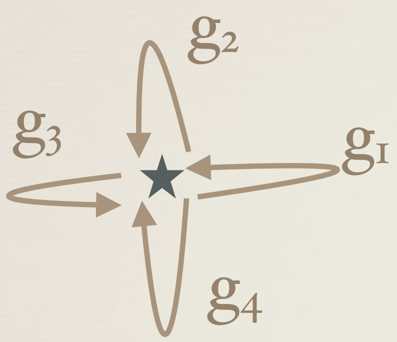
A. $\mathbb{C}(\mathbb{V}(R)) = R.$

B. $R = \bigcap_{i \in I} \mathbb{C}(\sigma_i)$ where σ_i ranges over all subobjects of the form $* \hookrightarrow \mathcal{I}(t)$.

Galois theory of field extensions

Fix $K \subseteq F \subseteq L$ Galois extensions of a field K.

Set



- T= $\mathrm{Aut}_K(L)$
- D= intermediate field extensions with homomorphisms fixing K.
- The functor I sends ★ to L and acts identically on arrows.

Simple calculations give:

$$\mathbb{C}(F) = \mathrm{Aut}_K(F)$$

$$\mathbb{V}(H) = L^H$$

The end

Thank you!