Representation of MV-algebras by regular ultrapowers of [0,1]

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Abstract

We present a *uniform* version of Di Nola Theorem, this enables to embed *all* MV-algebras of a bounded cardinality in an algebra of functions with values in a single non-standard ultrapower of the real interval $[0,1]$. This result also implies the existence, for any cardinal α, of a single MV-algebra in which all infinite MV-algebras of cardinality at most α embed. Recasting the above construction with *iterated ultrapowers*, we show how to construct such an algebra of values in a *definable* way, thus providing a sort of "canonical" set of values for the functional representation.

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1 Introduction

The main general tools in representation theory of MV-algebras are given by Chang representation Theorem [4], McNaughton Theorem and Di Nola representation Theorem [9]. The first gives a subdirect representation of all MV-algebras via linearly ordered MV-algebras (MV-chains). The second gives a characterisation of *free* MV-algebras as algebras of continuous, piece-wise linear functions with integer coefficients on [0,1]. Finally, the last describes MV-algebras as subalgebras of algebras of functions with values into a non-standard ultrapower of the MV-algebra $[0,1]$; in other words, given an arbitrary MV-algebra A, there exist an ultrapower $^*[0,1]$ and a set X such that A can be embedded into the MV-algebra of functions from X to $^*[0,1]$.

In the light of the latter functional representation, it is worth to stress that the class of *semisimple* MV-algebras can be characterised, up to isomorphism, as the class of subalgebras of the MV-algebras $[0,1]^X$ for some set X. Notably free

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MV-algebras are semisimple, whereas the Lindenbaum algebra of the infinite valued Łukasiewicz predicate logic is not [3].

In this paper we investigate how the *non-standard* representation of an MV-algebra A depends on the cardinality of A. For any infinite cardinal α, we prove that there exists an ultrapower U such that *all* MV-algebras of cardinality at most α are embedded in an MV-algebra of functions with values in U. In particular, we get a functional representation of all countable MV-algebras. Using the well-known Mundici's functor, the above results can be transferred to abelian lattice ordered groups with strong unit.

The main logical motivation to our investigation comes from the following considerations on the logic of perfect MV-algebras. This explanation would require many preliminary details on perfect MV-algebras and first order Łukasiewicz logic, which fall outside the scope of this article; we suggest [11, 1, 2] for all unexplained notions in the end of this section.

Let **Ł** denote the propositional Łukasiewicz logic [13] and let $\mathbf{Ł}_p$ be the extension of **Ł** with the axiom $(2\varphi^2) \leftrightarrow (2\varphi)^2$. In [2] it is proved that $\mathbf{Ł}_p$ is the logic of perfect MV-chains.

A similar result also holds for the first order Łukasiewicz logic, Ł$\forall$. Namely, let Ł$\forall_p$ be the extension of Ł$\forall$ with the axiom $(2\varphi^2) \leftrightarrow (2\varphi)^2$ and let $\mathbf{P}$ be the class of all perfect MV-chains, then a formula φ is provable in **Ł**$\forall_p$ if, and only if, it is valid in every $\mathbf{P}$-structure[1].

On the other hand the following characterisations also hold.

Proposition 1.1 ([1]). *Let* $\boldsymbol{K}$ *be the class of all ultrapowers of* $\Gamma(\mathbb{Z}\times_{lex}\mathbb{R},(1,0))$*, i.e., all perfect MV-chains of type* ${}^*\Gamma(\mathbb{Z}\times_{lex}\mathbb{R},(1,0))$.

- *The logic* $\boldsymbol{L}_p$ *is complete with respect to* $\boldsymbol{K}$
- *The logic* ***Ł***$\forall_p$ *is complete with respect to all* $\boldsymbol{K}$*-structures.*

When dealing with completeness results one may restrict to countable algebras. Since all the algebras from $\mathbf{K}$ are pairwise elementarily equivalent, then by the joint embedding property, there exists a perfect MV-algebra $U \in \mathbf{K}$ such that every countable algebra in $\mathbf{K}$ embeds into U. Since all the embeddings from any MV-algebra $A \in \mathbf{K}$ into U are elementary we get:

Theorem 1.2.

- *The logic* $\mathbf{L}_p$ *is complete with respect to* U.
- *The logic* ***Ł***$\forall_p$ *is complete with respect to all* U*-structures.*

The extension of such result to the Łukasiewicz logic is a challenging task which motivates, from the logical side, the study presented here.

To make the exposition self-contained and available for a reader with only fragmentary knowledge of ultrapowers and model theory, in addition to the reference, we will also give sketches of the proofs of some model theoretic results needed later.

[1] $\mathbf{K}$-structure are models in which the truth values range in some algebra in $\mathbf{K}$, for further details see [11]

2 Preliminaries

Definition 2.1. A structure $A = (A, \oplus, \neg, 0)$ is an **MV-algebra** if A satisfies the following equations, for every $x, y \in A$:

(i) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;	(ii) $x \oplus y = y \oplus x$;
(iii) $x \oplus 0 = x$;	(iv) $x \oplus \neg 0 = \neg 0$;
(v) $\neg\neg x = x$;	(vi) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Any MV-algebra is naturally endowed with a partial order given by $x \leq y$ if, and only if, $\neg x \oplus y = 1$; we call **MV-chain** any MV-algebra in which the order defined above is linear. For further details on MV-algebra the standard reference is [7].

Definition 2.2. Let A be an MV-algebra, a sequence $\mathbf{a} = \langle a_k : k \in \mathbb{Z} \rangle$ in A is **good** if the following properties are satisfied:

1. $a_k \oplus a_{k+1} = a_k$ for all $k \in \mathbb{Z}$,
2. there is some $n \in \mathbb{N}$ such that $a_k = 0$ for all $k \geq n$ and $a_k = 1$ for all $k \leq -n$.

One of the most surprising properties enjoyed by MV-algebras is their categorical equivalence with abelian lattice-ordered groups with strong unit (ℓu-group, for short) [14]. Indeed, given any ℓu-group $(G, +, 0, u)$ one may define an MV-algebra by considering its *unit interval*, $[0, u]$, endowed with the following operations (definable in the language of the ℓu-group):

$$x \oplus y = (x + y) \wedge u \quad \text{and} \quad \neg x = u - x$$

This MV-algebra is indicated by $\Gamma(G, u)$. Vice versa, let A be an MV-algebra and G_A be the set of all good sequences in A, then a structure of ℓu-group can be defined on G_A. The correspondence is such that if $A = \Gamma(G, u)$ then G_A is isomorphic to G, and vice versa. For further details see [7].

Lemma 2.3. *Let A be an MV-algebra and (G, u) an ℓu-group such that $A \cong \Gamma(G, u)$. Let α be an infinite cardinal then $|A| = \alpha$ if, and only if, $|G| = \alpha$.*

Proof. Suppose $|A| = \alpha$, with α an infinite cardinal. Let $\mathbf{a} = \langle a_i \rangle$ be a good sequence. Then there is an integer n_a such that $a_i = 1$ for all $i \leq -n_a$ and $a_i = 0$ for all $i \geq n_a$. Thus we can identify the good sequence $\mathbf{a}$ with a $2n_a + 1$ tuple, $\langle a_{-n_a}, a_{-n_a+1}, ..., a_{n_a-1}, a_{n_a} \rangle$. Call n_a the index of $\mathbf{a}$. Let S_n be the set of tuples of index n, then $|S_n| \leq \alpha^{2n+1}$. But $\alpha^{2n+1} = \alpha$. Thus $|S_n|$ has cardinality at most α. The set S of all good sequences is the union of all S_n. Hence S is $|S| \leq \alpha|\mathbb{N}|$, which is just α since $\mathbb{N}$ is denumerable. So the set of good sequences S has cardinality at most α. But since $|S_1| = |A| = \alpha$ the cardinality of S must be exactly α. Vice versa, if $|G| = \alpha$ then obviously $|A| \leq \alpha$. But if $|A| = \beta < \alpha$ then $|G_A| = \beta < \alpha$, since G_A is isomorphic to G. This gives a contradiction, hence $|G| = \alpha$. ∎

Lemma 2.4. *Let G be an abelian ℓ-group and α be an infinite cardinal such that $|G| = \alpha$. Then G can be embedded into an abelian divisible ℓ-group D_G such that $|D_G| = \alpha$*

Proof. First we show that any abelian group can be embedded in an abelian divisible group of cardinality α, where $\alpha = \max\{\aleph_0, |G|\}$. Indeed, let X be a generating set of G and let F be the free abelian group with basis X. Let H be a subgroup of F such that G is isomorphic to F/H. Let D be the direct product of $|X|$ copies of $\mathbb{Q}$. Since F is isomorphic to the direct product of $|X|$ copies of $\mathbb{Z}$ it can embedded in D, in a natural way. Hence G embeds in the abelian divisible group D/H. Moreover $|D/H| \leq |X| \cdot |\mathbb{Q}| \leq \alpha$ and obviously $|G| \leq |D/H|$, whence $|D/H| = \alpha$. Finally, the group D/H can be converted into an ℓ-group by taking as positive the elements h of D/H such that $nh \in G^+$ for some positive integer n. □

We recall some notions and results from model theory.

Definition 2.5. Let α be a cardinal. A proper filter D over I is said to be **α-regular** if there exists a set $E \subseteq D$ such that $|E| = \alpha$ and each $i \in I$ belongs to only finitely many $e \in E$.

For any set I of infinite cardinality α there exists an α-regular ultrafilter over I [6]. Given a cardinal α, let α^+ be the smallest cardinal greater than α. We let $\equiv$ stand for elementary equivalence, $\hookrightarrow$ for embeddability and $\hookrightarrow_{el.}$ for elementary embeddability.

Definition 2.6. Given a cardinal α, we say that a model $\mathfrak{A}$ is **α-universal** if and only if for every model $\mathfrak{B}$ we have:

$$\mathfrak{B} \equiv \mathfrak{A} \quad \text{and} \quad |\mathfrak{B}| < \alpha \quad \text{implies} \quad \mathfrak{B} \hookrightarrow_{el.} \mathfrak{A}.$$

Theorem 2.7 ([6])**.** *Let $|\mathcal{L}| \leq \alpha$ and D be an ultrafilter which is α-regular. Then, for every model $\mathfrak{A}$, the ultrapower $\prod_D \mathfrak{A}$ is α^+-universal.*

Proof. Let $E \subseteq D$ such that each $i \in I$ belongs to only finitely many $e \in E$ and $|E| = \alpha$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models such that $\mathfrak{A} \equiv \mathfrak{B}$ and $|\mathfrak{B}| \leq \alpha$. Consider $\Gamma_{\mathfrak{B}}$, the elementary diagram of $\mathfrak{B}$ in the expanded language $\mathcal{L}(\mathfrak{B})$ ($\mathcal{L}(\mathfrak{B})$ contains a new constant for any element in $\mathfrak{B}$). To prove that $\mathfrak{B}$ is elementarily embedded in $\prod_D \mathfrak{A}$ it suffices to find an expansion $(\prod_D \mathfrak{A}, a_b)_{b\in\mathfrak{B}}$ of the ultrapower $\prod_D \mathfrak{A}$, which is a model of $\Gamma_{\mathfrak{B}}$.
Since $|\Gamma_{\mathfrak{B}}| \leq \alpha$, there is a injective function $H : \Gamma_{\mathfrak{B}} \longrightarrow E$. Let us fix $i \in I$; since there are only finitely many $e \in E$ such that $i \in e$, we can consider the conjunction $\varphi \in \Gamma_{\mathfrak{B}}$ of all sentences ψ such that $i \in H(\psi)$. If the sentence φ contains parameters $c_1, ..., c_k$, it can be associated to an$\mathcal{L}$-sentence $\varphi' := \exists x_1 ... \exists x_k \varphi(c_1/x_1, ..., c_k/x_k)$, with $x_1, ..., x_k$ fresh variables; obviously if φ holds in $(\mathfrak{B}, b)_{b\in\mathfrak{B}}$ then φ' holds in $\mathfrak{B}$, and therefore φ' holds in $\mathfrak{A}$. This gives a way of building an expansion $(\mathfrak{A}, f_b(i))_{b\in\mathfrak{B}}$ of $\mathfrak{A}$ to $\mathcal{L}(\mathfrak{B})$, which is a model of φ. Notice that $f_b \in A^I$ is such that for all $i \in I$ and $\psi \in \Gamma_{\mathfrak{B}}$,

$$i \in H(\psi) \text{ implies } (\mathfrak{A}, f_b(i))_{b\in\mathfrak{B}} \models \psi.$$

Moreover, for each $\psi \in \Gamma_{\mathfrak{B}}$, $H(\psi) \in D$, whence by Łoś Theorem $\prod_D(\mathfrak{A}, f_b(i))_{b\in\mathfrak{B}} \models \psi$. So $\prod_D(\mathfrak{A}, a_b)_{b\in\mathfrak{B}}$ with $a_b = (f_b)_D$ is the required expansion. □

3 MV-algebras of infinite cardinality

Let $\mathcal{L}_{MV} = \{0, \neg, \oplus\}$ be the language of MV-algebras and $\mathcal{L}_{lg} = \{0, -, +, \vee, \wedge, \leq\}$ be the language of ℓ-groups.

The following is a simple generalisation of a result contained in [5].

Theorem 3.1. *For any sentence σ in $\mathcal{L}_{MV}$ there is a formula with only one free variable $\widehat{\sigma}(v)$ in $\mathcal{L}_{lg}$ such that for any MV-algebra A we have:*

$$A \models \sigma \quad \text{if, and only if,} \quad G \models \widehat{\sigma}[u],$$

for any abelian ℓ-group G and $u > 0$ in G such that $A \equiv \Gamma(G, u)$.

Recall also that:

Theorem 3.2 ([15, 3.1.2])**.** *Any non-trivial divisible totally ordered ℓ-group is elementarily equivalent to the additive group $\mathbb{R}$ of real numbers.*

The MV-algebra counterpart of divisible groups are called **divisible MV-algebras** [10], the functor Γ remains a categorical equivalence when restricted to those categories. From Theorem 3.1 and Theorem 3.2, it can be proved:

Theorem 3.3. *Any non-trivial divisible MV-chain is elementarily equivalent to* $\Gamma(\mathbb{R}, 1) = [0, 1]$.

Theorem 3.4. *Let α be an infinite cardinal and A be an MV-chain such that $|A| = \alpha$. A can be embedded into an ultrapower of the MV-algebra* $[0, 1]$ *via an ultrafilter α-regular over α which does not depend on A.*

Proof. Let A be an infinite MV-chain such that $|A| = \alpha$ and $A \cong \Gamma(G, u)$. Then, by Lemma 2.3, G is an ordered abelian group with strong unit u and $|G| = \alpha$. Hence (G, u) can be embedded into a divisible ordered group D_G with strong unit u_D; in addition, by Lemma 2.4, $|D_G| = \alpha$. Now let $A_d = \Gamma(D_G, u_D)$: then A embeds in A_d and A_d is a divisible MV-algebra; so by Theorem 3.3, A_d is elementarily equivalent to $[0, 1]$.

Let F be a α-regular ultrafilter over α; then, by Theorem 2.7, $\prod_F[0, 1]$ is α^+-universal, hence $A_d \hookrightarrow \prod_F[0, 1]$. This proves that A can be embedded into the ultrapower $\prod_F[0, 1]$. □

Some comments are in order here. First of all, we would like to mention that, as the anonymous referee suggests, one can avoid the use of α-regular ultrafilters and prove the existence of such an ultrapower by using the joint embedding property for divisible MV-algebras. Indeed one can order all divisible MV-algebra of a bounded cardinality and then, starting from the first two, repeatedly embed the successive in the list together with the one previously obtained. This, together with Lemma 2.3, Lemma 2.4 and Theorem 3.3, gives the above result. Secondly, Theorem 3.4 may seem to fit in the general theory of Jónsson classes [8], however the existence of a homogeneous-universal structure for a Jónsson class is guaranteed only for a few particular cardinals. This makes our result stronger than what can be directly obtained from the theory of Jónsson classes.

The following fact, which is part of the classical literature on the subject, gives a sharp estimation of the cardinality of the target algebra.

Theorem 3.5 ([6]). *Let F be a α-regular ultrafilter of α, with α infinite cardinal, then $|\prod_F A| = |A|^\alpha$.*

Let us recall that an **ideal** of an MV-algebra is a non-empty downward-closed set, closed under $\oplus$. An ideal I is called **prime** if whenever $x \wedge y \in I$ then either $x \in I$ or $y \in I$. Given an MV-algebra A it is customary to denote by $Spec(A)$ the set of its prime ideals.

Theorem 3.6. *Let A be an MV-algebra such that $|A| = \alpha$, with α an infinite cardinal. Then there exists a set X such that A can be embedded into an MV-algebra of functions from X to an ultrapower of the MV-algebra $[0,1]$ via an α-regular ultrafilter over α which does not depend on A.*

Proof. Let A be an infinite MV-algebra of cardinality α. Then by Chang's representation theorem we have:

$$A \hookrightarrow \prod_{P \in Spec(A)} A/P.$$

Obviously for every prime ideal P of A, the quotient is such that $|A/P| \leq \alpha$.

Let F be an ultrafilter α-regular over α. By Theorem 2.7, the ultrapower $\prod_F [0,1]$ of the MV-algebra $[0,1]$ is α^+-universal. Then for every $P \in Spec(A)$ the MV-chain A/P can be embedded into $\prod_F [0,1]$. Indeed the ultrafilter F is independent of A/P. Hence A can be embedded into $(\prod_F [0,1])^{Spec(A)}$. □

For any cardinal α, the above theorem gives an MV-algebra of *values* U such that any MV-algebra of cardinality at most α can be represented as an algebra of functions with value in U. Now it seems natural to ask further, whether there exists a single MV-algebra at all, in which all MV-algebras of cardinality at most α embed.

Corollary 3.7. *For any infinite cardinal α there exists an MV-algebra of functions A such that any MV-algebra of infinite cardinality at most α embeds in A.*

Proof. In the proof above only X depends on A, so it is sufficient to remove this dependency. Let MV_α be the class of all MV-algebras of cardinality at most α and let K_α be the set of the cardinalities of $Spec(A)$ for $A \in MV_\alpha$, i.e.

$$K_\alpha = \{\beta \mid \exists A \in MV_\alpha \text{ with } |Spec(A)| = \beta\}.$$

Let $A \in K$ and $|Spec(A)| = \beta$, then by the subdirect representation and Theorem 3.4 $A \hookrightarrow \prod_F [0,1]^\beta$.

Now let $\gamma = \sup K_\alpha$, then each $\prod_F [0,1]^\beta$ with $\beta < \gamma$ embeds in $\prod_F [0,1]^\gamma$ via the function

$$\Phi(f)(\xi) = \begin{cases} f(\xi) & \text{if } \xi < \beta \\ f(0) & \beta \leq \xi < \gamma. \end{cases}$$

So each $A \in MV_\alpha$ embeds in $\prod_F [0,1]^\gamma$ through the composition of the two embeddings above. □

4 On canonical representations

In this section we refine the previous embedding theorems by using certain *definable*[2] structures, rather than usual (nondefinable) ultrapowers. These structures are *iterated ultrapowers* in the sense of [6, Section 6.5]; although we will define iterated ultrapowers in this paper, we refer to this last volume for a more detailed exposition of the argument and for all unproved claims which follow.

According to [6], an iterated ultrapower is a structure which can be obtained out of a linearly ordered set of ultrapowers of a given base structure in such a way that all these ultrapowers are elementarily embedded in it. We outline here the formal construction of iterated ultrapowers (taken from [6] with a few simplifications), starting with some preliminary work of a combinatorial flavour.

Let A be a first order structure, B be a set, $(Y, <)$ be a linear order, and $D = (D_y, y \in Y)$ be a linearly ordered sequence of ultrafilters on B, indexed by Y, and possibly containing repetitions.

Let $K = B^Y$ be the set of all functions from Y to B. Let Z be a subset of Y. We say that a function f with domain K **lives on** Z if, for every function $i \in K$, $f(i)$ depends only on $i|_Z$; and that a subset of K **lives on** Z if its characteristic function lives on Z.

In the following let Z range over *finite* subsets of Y. Let B^Z the set of all functions from Z to B. To each Z we can associate an ultrafilter D_Z on B^Z as follows (our definition is slightly different from [6], but defines the same object).

Let $Z = y_1 < \ldots < y_n$ be the order on Z induced by Y. Any function from Z to B (viewed as a set of pairs) can be written as $\{(y_1, \alpha_1), \ldots, (y_n, \alpha_n)\}$ for some $\alpha_1, \ldots, \alpha_n \in B$. Now let

$$D_Z = \{s \subseteq B^Z : D_{y_1}\alpha_1 \ldots D_{y_n}\alpha_n.\{(y_1, \alpha_1), \ldots, (y_n, \alpha_n)\} \in s\},$$

where $D_y\alpha.\phi(\alpha)$ means $\{\alpha : \phi(\alpha)\} \in D_y$.

The set D_Z is an ultrafilter on B^Z, and coincides with the usual finitary product ultrafilter $D_{y_1} \times \ldots \times D_{y_n}$.

Now we can consider the set

$$E(D) = \{s \subseteq K : \exists Z.\ s \text{ lives on } Z \text{ and } (s \downarrow Z) \in D_Z\},$$

[2]More precisely we mean definable in ZFC.

where $s \downarrow Z$ is the set of all restrictions to Z of the members of s.

We note that the subsets of K living on some finite subset of Y form a Boolean algebra S, and the set $E(D)$ is an ultrafilter in S. $E(D)$ can be considered as an infinitary product of the ultrafilters D_y (although it is *not* an ultrafilter on K as one could expect).

Now we are ready to define iterated ultrapowers, and we do it in analogy with [6].

Definition 4.1. Let A be a first order structure, B be a set, and D be a linearly ordered sequence of ultrafilters on B indexed by the linear order $(Y, <)$. The **iterated ultrapower of** A **on** D, denoted $\Pi_D A$, is a first order structure over the same language as A. The domain of $\Pi_D A$ is the set of all functions f from $K(= B^Y)$ to A which live on some finite subset of Y, modulo the equivalence $=_D$ given by:

$$f =_D g \text{ if, and only if, } \{i \in K : f(i) = g(i)\} \in E(D).$$

Every n-ary predicate symbol R of the language of A is interpreted in $\Pi_D A$ by the following relation R':

$$R'([f_1], \ldots, [f_n]) \text{ if, and only if, } \{i \in K : R^A(f_1(i), \ldots, f_n(i))\} \in E(D),$$

where $[f]$ is the equivalence class of f modulo $=_D$, and R^A is the interpretation of R in A. Likewise, every n-ary function symbol G is interpreted by the following function G':

$$G'([f_1], \ldots, [f_n]) = [g] \text{ if, and only if, } \{i \in K : G^A(f_1(i), \ldots, f_n(i)) = g(i)\} \in E(D),$$

where G^A is the interpretation of G in A; and every constant symbol c is interpreted by letting $c'(i) = c^A$ for every $i \in K$, where c^A is the interpretation of c in A.

Note that the definitions above do not depend on the choice of representatives in the classes, as usual when defining ultraproducts or similar.

An embedding property of iterated ultrapowers is:

Lemma 4.2 ([6, Prop. 6.5.9])**.** *For every $y \in Y$, $\Pi_{D_y} A$ embeds elementarily in $\Pi_D A$.*

So far we have a general notion of iterated ultrapower. In the present situation, we start from a base structure A and a set B (where we can suppose that B is a cardinal α), and we would like to build a *definable* iterated ultrapower of all ultrapowers of A modulo ultrafilters on α.

The problem is that there is no available definable linear ordering of these ultrapowers. However, we can use the same stratagem of [12], developed to find a definable model of nonstandard analysis. Namely, we index, with repetitions, ultrapowers by using a set of indexes which has a natural linear ordering, definable in α.

The formal construction is as follows.

Lemma 4.3. *Let A be a first order structure and let α be an infinite cardinal. There is an iterated ultrapower $\Pi_D A$ of A, definable in A and α, where all ultrapowers of A over α embed elementarily.*

Proof. Let $\mathcal{P}(\alpha)$ be the powerset of α. We note that $\mathcal{P}(\alpha)$ has a natural "lexicographic" linear order: given $E, F \subseteq \alpha$, we let $E < F$ if E and F are different, and the least element of α where E and F differ belongs to F.

Moreover, let $|\mathcal{P}(\alpha)|$ be the cardinality of $\mathcal{P}(\alpha)$, that is, the least ordinal having a bijection with $\mathcal{P}(\alpha)$.

Let Y be the set of all maps $y : |\mathcal{P}(\alpha)| \to \mathcal{P}(\alpha)$ such that the image of y is an ultrafilter on α. Note that every ultrafilter on α appears as image of some (actually infinitely many) elements of Y.

The set Y is totally ordered by setting $y < y'$ if there is an ordinal $\xi < |\mathcal{P}(\alpha)|$ such that $y|_\xi = y'|_\xi$ (that is, y and y' coincide on all the ordinals less than ξ) and $y(\xi) < y'(\xi)$ in the lexicographic order of $P(\alpha)$.

For every $y \in Y$, let D_y be the ultrafilter on α associated to y, i.e. the image of y. Let D_α be the resulting indexed family of ultrafilters on α. Then $D = D_\alpha$ has the required properties. In fact, $\Pi_{D_\alpha} A$ is definable in A and α by construction, and every $\Pi_{D_y} A$ embeds elementarily in $\Pi_{D_\alpha} A$ by the previous lemma. □

Lemma 4.3 above gives us:

Theorem 4.4. *For every infinite cardinal α there is an iterated ultrapower Π_α of $[0,1]$, definable in α, where every MV-chain of cardinality α embeds.*

Proof. By Di Nola Theorem for MV-chains, for every infinite cardinal α and for every MV-chain A of cardinality α, there is an ultrapower of $[0,1]$, modulo an ultrafilter on α, where A embeds. Now let D_α be the sequence of ultrafilters on α built in the previous lemma.

By construction, $\Pi_{D_\alpha}[0,1]$ is an iterated ultrapower definable in α (the base structure is $[0,1]$ which is definable itself), where *all* ultrapowers of $[0,1]$ modulo ultrafilters on α embed; hence, all MV-chains of cardinality α embed in $\Pi_{D_\alpha}[0,1]$ as well, and we can take $\Pi_\alpha = \Pi_{D_\alpha}[0,1]$. □

Corollary 4.5. *For every infinite cardinal α there is an iterated ultrapower Π_α of $[0,1]$, definable in α, such that every MV-algebra of cardinality α embeds in a power of Π_α.*

Proof. Take as Π_α the iterated ultrapower of the previous theorem. By the Chang's embedding, we have for every MV-algebra A of cardinality α:

$$A \hookrightarrow \prod_{P \in Spec(A)} \frac{A}{P} \hookrightarrow \prod_{P \in Spec(A)} \Pi_\alpha = (\Pi_\alpha)^{Spec(A)}.$$

□

It should be noted that also in this case the proof of Corollary 3.7 works, yielding a definable MV-algebra in which all elements of MV_α embed (yet, not in a definable way, for if A is an MV algebra, there is no available definable isomorphism between $\prod_\alpha^{Spec(A)}$ and $\prod_\alpha^{|Spec(A)|}$).

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