Many-Valued Logics (Autumn 2013)

Second homework assignment

- Deadline: 19 September — at the beginning of class.
- Grading is from 0 to 100 points; you get 10 points for free.
- Success!

Exercise 1. (Embedding into the real interval)
Consider the BL-algebra $\langle C, *, \Rightarrow, \land, \lor, 0, 1 \rangle$ where

1. $C = \{a_n \mid n \in \mathbb{N}\} \cup \{b_n \mid n \in \mathbb{N}\}$ where all $a_i$ and $b_j$ are different, and $a_0 = 1$ and $b_0 = 0$.

2. The order is given by
   - (a) $\forall m, n \in \mathbb{N} \ b_m < a_n$,
   - (b) $\forall m, n \in \mathbb{N} \ a_m < a_n$ if, and only if, $m > n$,
   - (c) $\forall m, n \in \mathbb{N} \ b_m < b_n$ if, and only if, $m < n$.

3. $*$ is defined as
   - (a) $\forall m, n \in \mathbb{N} \ b_m * b_n = 0$,
   - (b) $\forall m, n \in \mathbb{N} \ a_m * a_n = a_{m+n}$,
   - (c) $\forall m, n \in \mathbb{N} \ a_m * b_n = \begin{cases} b_{n-m} & \text{if } m \leq n \\ 0 & \text{otherwise.} \end{cases}$

4. $\Rightarrow$ is the residuum of $*$. The operations $\land$ and $\lor$ are, respectively, minimum and maximum w.r.t. the order defined above.

Prove that $C$ cannot be homomorphically embedded in any standard BL-algebra.

Exercise 2. (Finitely generated BL-algebras)
Recall that an algebra $A$ is called finitely generated if there exists a finite set $\{a_1, ..., a_n\} \subseteq A$ such that for any $a \in A$ there exists a term $t(x_1, ..., x_n)$ in the language of $A$ such that $a = t(a_1, ..., a_n)$.

Prove that any linearly ordered finitely generated BL-algebra is the ordinal sum of finitely many irreducible components.

Exercise 3. (Lukasiewicz t-norm)
Prove that the Lukasiewicz t-norm is the only continuous t-norm with a continuous residuum.