Many-Valued Logics (Autumn 2013)

Sixth homework assignment

• Deadline: 17 October — at the beginning of class.
• Grading is from 0 to 100 points; you get 10 points for free.
• Success!

Exercise 1. (Γ functor) 30 pt
Consider the MV-algebra \( \langle C, \oplus, \neg, 0 \rangle \) where

1. \( C = \{a_n \mid n \in \mathbb{N}\} \cup \{b_n \mid n \in \mathbb{N}\} \) where all \( a_i \) and \( b_j \) are different, and \( a_0 = 1 \) and \( b_0 = 0 \).

2. The order is given by
   - (a) \( \forall m, n \in \mathbb{N} \quad b_m < a_n \),
   - (b) \( \forall m, n \in \mathbb{N} \quad a_m < a_n \) if, and only if, \( m > n \),
   - (c) \( \forall m, n \in \mathbb{N} \quad b_m < b_n \) if, and only if, \( m < n \).

3. \( \oplus \) is defined as
   - (a) \( \forall m, n \in \mathbb{N} \quad b_m \oplus b_n = b_{m+n} \),
   - (b) \( \forall m, n \in \mathbb{N} \quad a_m \oplus a_n = a_0 \),
   - (c) \( \forall m, n \in \mathbb{N} \quad a_m \oplus b_n = \begin{cases} a_{m-n} & \text{if } n \leq m \\ 1 & \text{otherwise.} \end{cases} \)

4. \( \neg a_n = b_n \) and \( \neg b_m = a_m \).

Describe the unital \( \ell \)-group \( (G, u) \) such that \( C = \Gamma(G, u) \).

(Hint: a nice description can be given using the lexicographic products of \( \ell \)-groups: \( G \times F = (G \times F, +, -, \leq, 0) \) where \( +, -, 0 \) are defined as in the direct product and \( (x_1, x_2) \leq (y_1, y_2) \) iff either \( x_1 < y_1 \), or \( x_1 = y_1 \) and \( x_2 \leq y_2 \)).

Exercise 2. (Standard MV-algebra) 30 pt
Theorem. Let \( (H, 1) \) and \( (G, 1) \) be sub-\( u\ell \)-groups of \( (\mathbb{R}, +, -, \leq, 0) \). There exists at most one \( u\ell \)-homomorphism between \( H \) and \( G \) and when it exists it must be the identity.

Prove that there is only one isomorphism form the standard MV-algebra \([0,1]\) into itself.
Exercise 3. (Standard completeness of Lukasiewicz logic)
Prove that an equation is true in all MV-algebras if, and only if, it is true in the standard
MV-algebra $[0,1]$. (Hint: at some point you may want to use that (i) every linearly ordered
abelian group embeds into a divisible linearly ordered abelian group, and that (ii) the theory
of linearly ordered divisible abelian groups is complete. An abelian group $(G, +, - , 0)$ is
divisible if, for every positive integer $n$ and every $g \in G$, there exists $y \in G$ such that $ny = g$)