Many-Valued Logics (Autumn 2013)

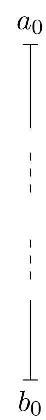
Sixth homework assignment

- Deadline: 17 October — at the **beginning** of class.
- Grading is from 0 to 100 points; you get 10 points for free.
- Success!

30 pt

Exercise 1. (Γ functor)
Consider the MV-algebra $\langle C, \oplus, \neg, 0\rangle$ where

1. $C = \{a_n \mid n \in \mathbb{N}\} \bigcup \{b_n \mid n \in \mathbb{N}\}$ where all a_i and b_j are different, and $a_0 = 1$ and $b_0 = 0$.
2. The order is given by
   (a) $\forall m, n \in \mathbb{N}\ b_m < a_n$,
   (b) $\forall m, n \in \mathbb{N}\ a_m < a_n$ if, and only if, $m > n$,
   (c) $\forall m, n \in \mathbb{N}\ b_m < b_n$ if, and only if, $m < n$.



3. $\oplus$ is defined as
   (a) $\forall m, n \in \mathbb{N}\ b_m \oplus b_n = b_{m+n}$,
   (b) $\forall m, n \in \mathbb{N}\ a_m \oplus a_n = a_0$,
   (c) $\forall m, n \in \mathbb{N}\ a_m \oplus b_n = \begin{cases} a_{m-n} & \text{if } n \le m \\ 1 & \text{otherwise.} \end{cases}$
4. $\neg a_n = b_n$ and $\neg b_m = a_m$.

Describe the unital ℓ-group (G, u) such that $C = \Gamma(G, u)$.
(Hint: a nice description can be given using the *lexicoraphic products of ℓ-groups*: $G \overrightarrow{\times} F = (G \times F, +, -, \le, 0)$ where $+, -, 0$ are defined as in the direct product and $(x_1, x_2) \le (y_1, y_2)$ iff either $x_1 < y_1$, or $x_1 = y_1$ and $x_2 \le y_2$.)

30 pt

Exercise 2. (Standard MV-algebra)
Theorem. Let $(H, 1)$ and $(G, 1)$ be sub-$u\ell$-groups of $\langle \mathbb{R}, +, -, \le, 0\rangle$. There exists at most one $u\ell$-homomorphism between H and G and when it exists it must be the identity.

Prove that there is only one isomorphism form the standard MV-algebra [0,1] into itself.

30 pt

Exercise 3. (Standard completeness of Łukasiewicz logic)
Prove that an equation is true in all MV-algebras if, and only if, it is true in the standard MV-algebra [0,1]. (Hint: at some point you may want to use that (*i*) every linearly ordered abelian group embeds into a *divisible* linearly ordered abelian group, and that (*ii*) the theory of linearly ordered divisible abelian groups is complete. An abelian group $(G, +, -, 0)$ is divisible if, for every positive integer n and every $g \in G$, there exists $y \in G$ such that $ny = g$)