Lecture 9 16/4/24

Marta : Prime ideals
Matteo : Projective MV-algebras
Sebast. : States on ℓ-group.

Teo 1 MV-algebras are categorically equivalent to ℓ-groups with strong units (unital ℓ-groups)

Teo 2 Semi-simple MV-algebras are dually equivalent to Tychonoff spaces (Compact Hausdorff spaces embedded in some Tychonoff cube $[0,1]^k$) and $\mathbb{Z}$-maps between them (cont. piece-wise affinely linear maps with integer coefficients)

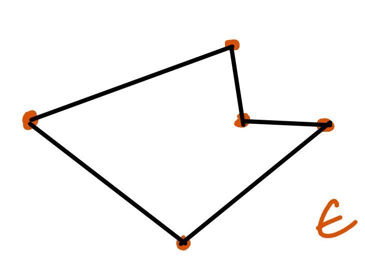
Cor 1 Finitely presented (hence semi-simple) MV-algebras are dually equivalent to the category of rational polyhedra (finite unions of convex hulls of rational points) and $\mathbb{Z}$-maps between them

Theo 3 Semi-simple (Abelian) ℓ-groups are dually equivalent to the category of closed cones in $\mathbb{R}^k$ (positive hulls in $\mathbb{R}^k$) and cont. piece-wise homogeneous maps with integer coefficients.

Cor 2 Finitely presented ℓ-groups are dually equivalent to the category of rational cones in $\mathbb{R}^n$ (positive hulls of rational vectors in $\mathbb{R}^n$) and cont. piece-wise homogeneous maps with integer coefficients.

finitely presented MV-algebras

$\bullet \in \mathbb{Q}^n$

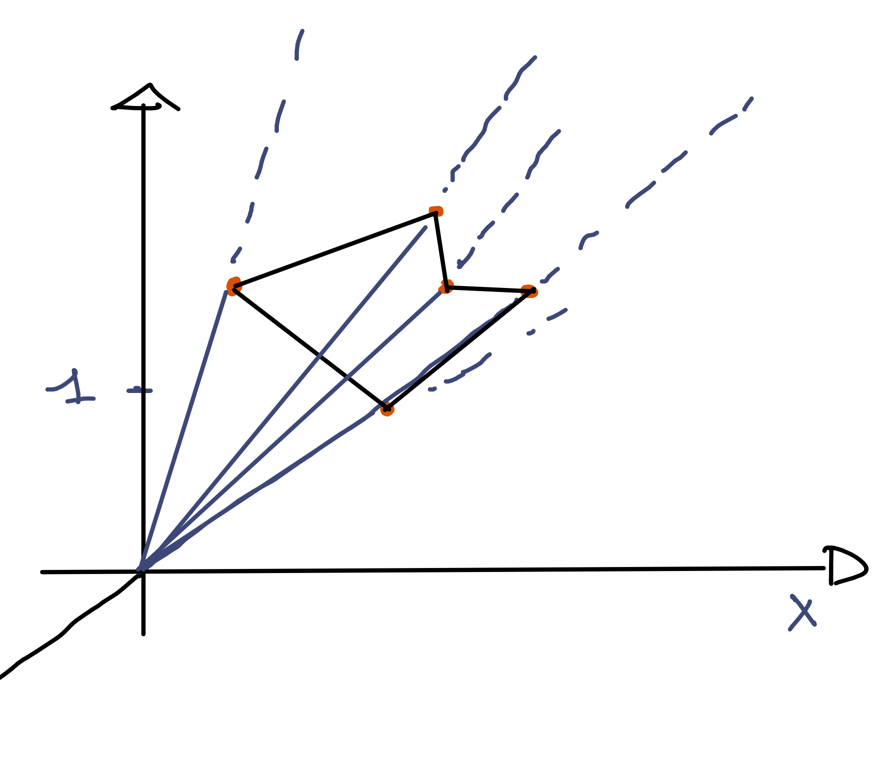


$\in \mathbb{Q}^n \rightsquigarrow \in \mathbb{Z}^{n+1}$

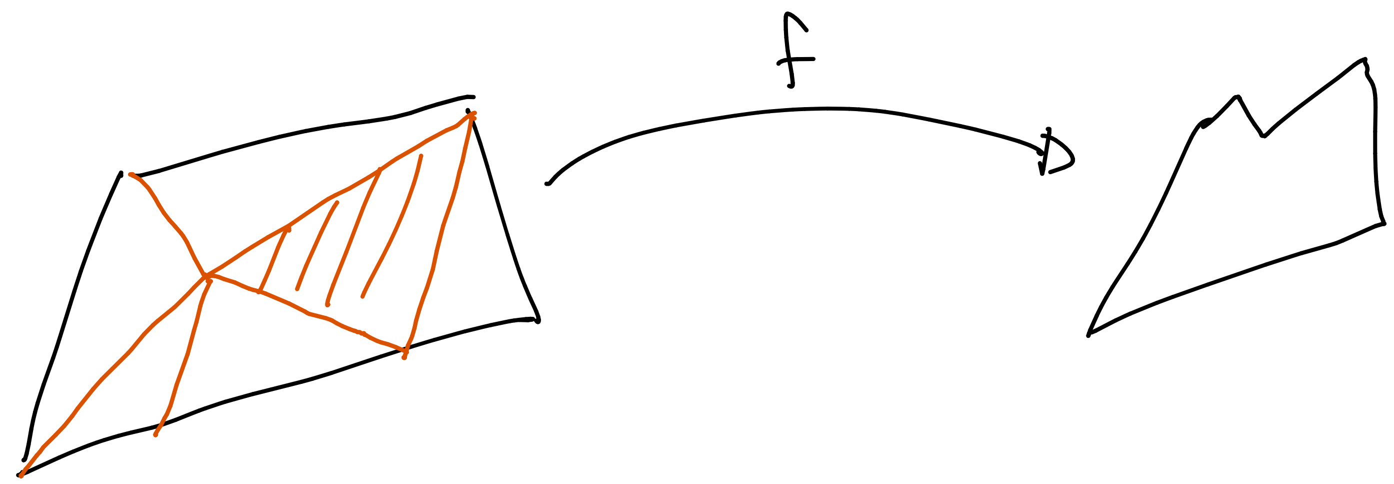
$\bar{q} := (q_1, q_2, \dots, q_n) \longmapsto \tilde{q} := d \cdot (q_1, \dots, q_n, 1)$

$d = \mathrm{lcd}(\bar{q})$

homogeneous correspond.



Notice that by def. a piecewise linear / homogeneous map f is a map from $P \to Q$ s.t. there is a "decomposition" of P in subobjects such that f is linear / homogeneous on each subobject



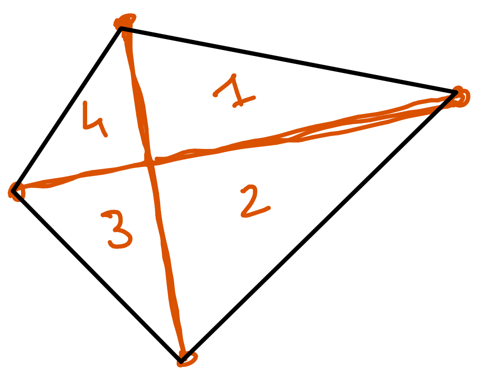
A polyhedron / polyhedral cone is called simplicial if it is the convex / positive hull of affinely / linear independent points

A simplicial complex is a set $\mathcal{C}$ of simplicial polyhedra / poly. cones s.t.

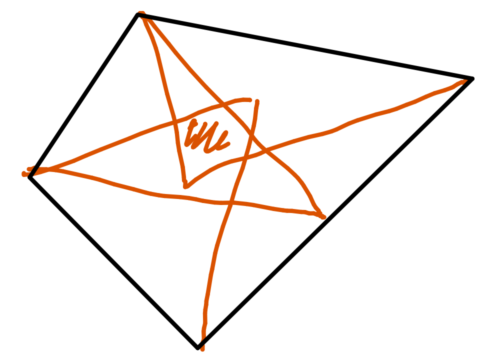
1) If $A, B \in \mathcal{C}$ then $A \cap B$ is a face of A and B

2) If $A \in \mathcal{C}$ all "faces" of A are also in $\mathcal{C}$

A simplicial complex



Not a simplicial complex.



We call a (simplicial) triangulation of P any simplicial complex $\mathcal{C}$ such that $|\mathcal{C}| := \bigcup \mathcal{C}$ is equal to P

the realization of $\mathcal{C}$

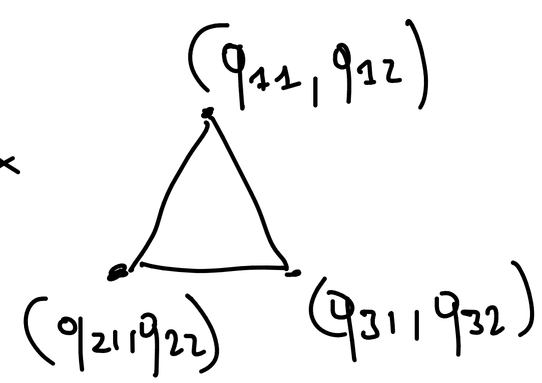
A combinatorial map between simplicial complexes $\mathcal{C}, \mathcal{D}$ is a map that respects the adjeciancy of $\mathcal{C}$

<u>Theor</u> Any combinatorial map between simplicial complexes $\mathcal{C}$ and $\mathcal{D}$ extends to a piece-wise linear/homogeneous map between the realization $|\mathcal{C}|$ and $|\mathcal{D}|$.

Theorem Given any PL-map $f : P \to Q$ one can find triangulations of P such that f is linear / homogeneous on any simplex of the triangulation.

When dealing with ℓ-groups we need also to take into account the integer coefficients.

simplex



$$\xrightarrow{\text{homog.}} \quad S := \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix}$$

$i = 1,2$

$z_{ji} := q_{ji} \,\mathrm{lcm}(\mathrm{den}\, q_{j1}, \mathrm{den}\, q_{j2})$

$z_{j3} := \mathrm{lcm}(\mathrm{den}\, q_{j1}, \mathrm{den}\, q_{j2})$

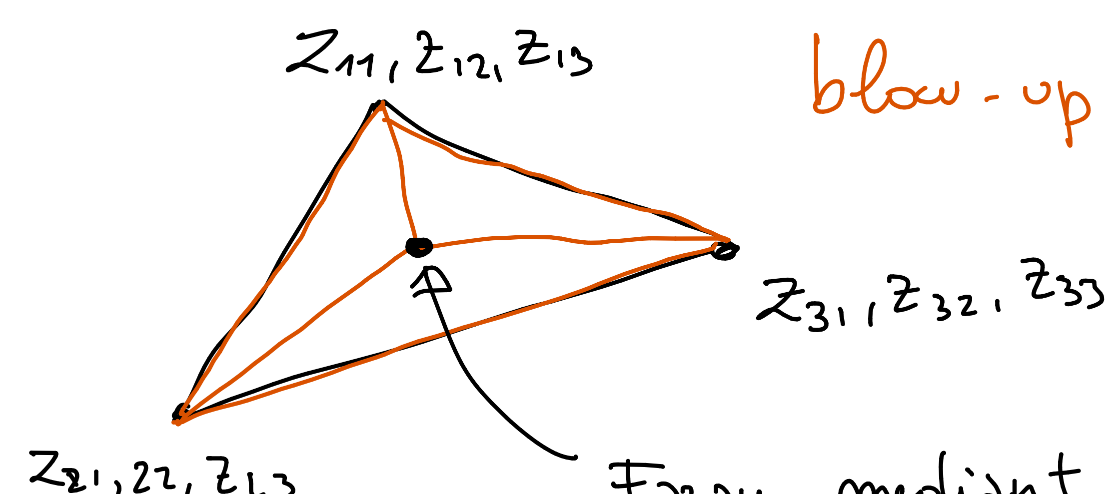
$A \cdot S = T \quad \to \quad A = T \cdot S^{-1}$ | can do this when $\det S = \pm 1$

We call a simplex S unimodular (regular) if $\det(S) = \pm 1$

We call a simplicial complex $\mathcal{C}$ unimodular or regular if all its simplices are unimodular (regular)

Theorem Let P, Q be rational polyhedra / poly cones and T, S be rational triangulations of P and Q, respectively. Suppose that T is unimodular and f is a combinatorial map from T to S such that $\operatorname{den} f(v) \mid \operatorname{den} v$ for any (rational) vertice of T then f extends to a $\mathbb{Z}$-map from P into Q.

Vice versa, if $g : P \to Q$ is a $\mathbb{Z}$-map then one can find triangulations of P and Q satisfying the above condition such that g is linear / homogeneous on each simplex of the triangulation.



Faray mediant of the vertices
$(z_{11} + z_{21} + z_{31},\ z_{12} + z_{22} + z_{32},\ z_{31} + z_{32} + z_{33})$

Lecture 10 18/4/24

(Beynon '77) Finitely generated projective ℓ-groups are exactly the finitely presented ones.

Def In a category $\mathcal{C}$ an object P is called projective if

$$\begin{tikzcd} & B \arrow[d, two heads, "\forall"] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, "\forall"'] & A \end{tikzcd}$$

In a variety of algebras projective objects are exactly the "retractions" of free algebras.

A retraction $r: X \to Y$ is a map for which there exists $s: Y \to X$ s.t. (section)

$id = r \circ s : Y \to Y$

It is easy to see that free objects are projective

$$\begin{tikzcd} & B \arrow[d, two heads, "g"] & \exists b \arrow[d, "g"] \\ \mathcal{F}(X) \arrow[ur, dashed, "h"] \arrow[r, "f"'] & A & \\ {[t(x)]} \arrow[rr, mapsto] & & a \end{tikzcd}$$

Just choose any b s.t. $g(b) = a$ and define $h([t(x)]) = b$.

It is then easy also to see that retracts of projective objects are again projective. Thus all retracts of free objects are projective.

Vice versa, let P be a projective object. There exists a set X s.t. $F(X) \twoheadrightarrow P$, consider

$$
\begin{tikzcd}
& F(X) \arrow[d, two heads, "r"] \\
P \arrow[ur, dashed, "\exists s"] \arrow[r, "id"'] & P
\end{tikzcd}
\qquad r \circ s = id
$$

Therefore P is a retract of $F(X)$

To prove the statement of the theorem notice that if G is projective and finitely generated, then it's a retract of a finitely generated free algebra $\mathcal{F}(x_1 \ldots x_n)$

$$
\begin{aligned}
\mathcal{F}(x_1 \ldots x_n) &\overset{r}{\twoheadrightarrow} P \overset{s}{\hookrightarrow} \mathcal{F}(x_1 \ldots x_n) \\
x_i &\longmapsto p_i = [t(x_1 \ldots x_n)]
\end{aligned}
$$

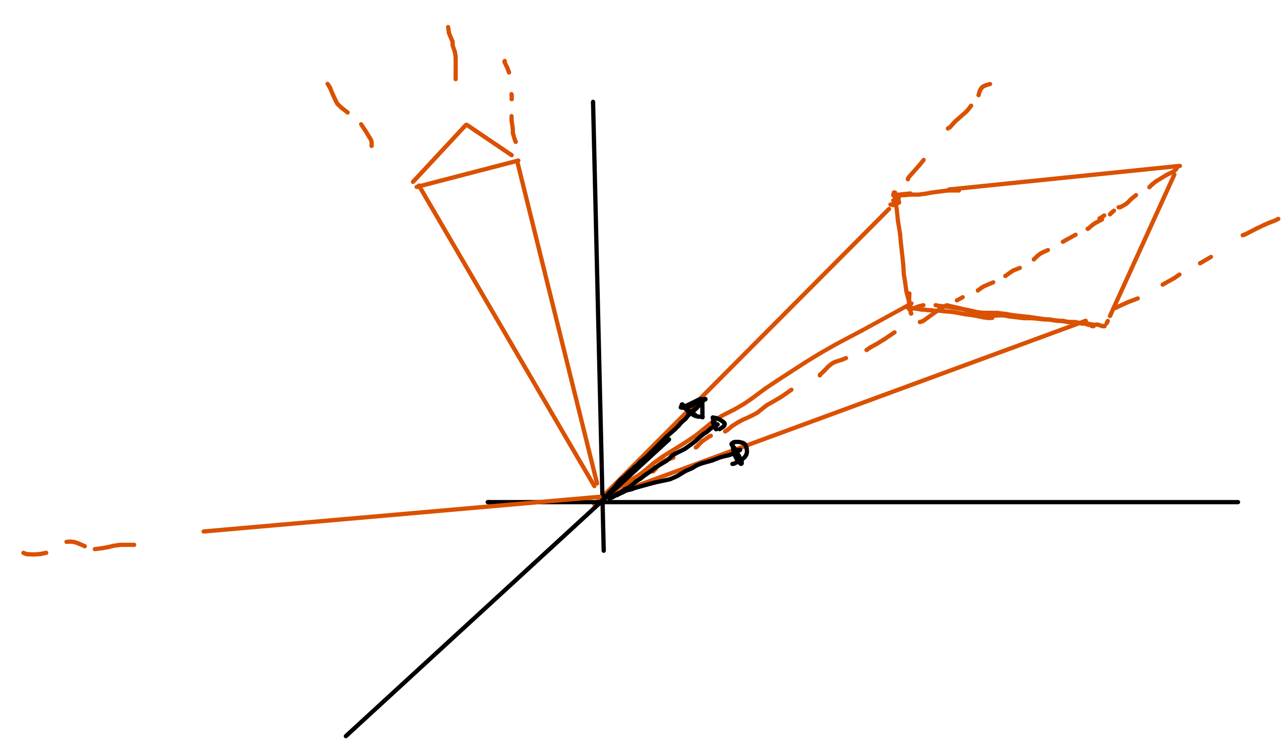
then the congruence generated by the pairs (x_i, p_i) presents P

This proves that any finitely generated projective object in a variety is finitely presented.

For the other direction we use the duality with rational polyhedral cones. Let G be a f.p. ℓ-group.

$$\begin{tikzcd}
G \arrow[r, hook] \arrow[d, Rightarrow] \arrow[dr, phantom, "\text{duality}"] & \mathcal{F}_\ell(x_1 \dots x_n) \arrow[r, two heads] \arrow[d, Rightarrow] \arrow[dr, phantom, "\text{duality}"] & G \arrow[d, Rightarrow] \\
C & \mathbb{R}^n \arrow[l, two heads] & C \arrow[l, hook']
\end{tikzcd}$$

In order to complete the proof of the theorem we need to find a retraction of $\mathbb{R}^n \twoheadrightarrow C$ for any rational polyhedral cone C



1 We start with two triangulations T for C and S for the whole $\mathbb{R}^n$.

2 We refine S to S' by blow up's to include T.

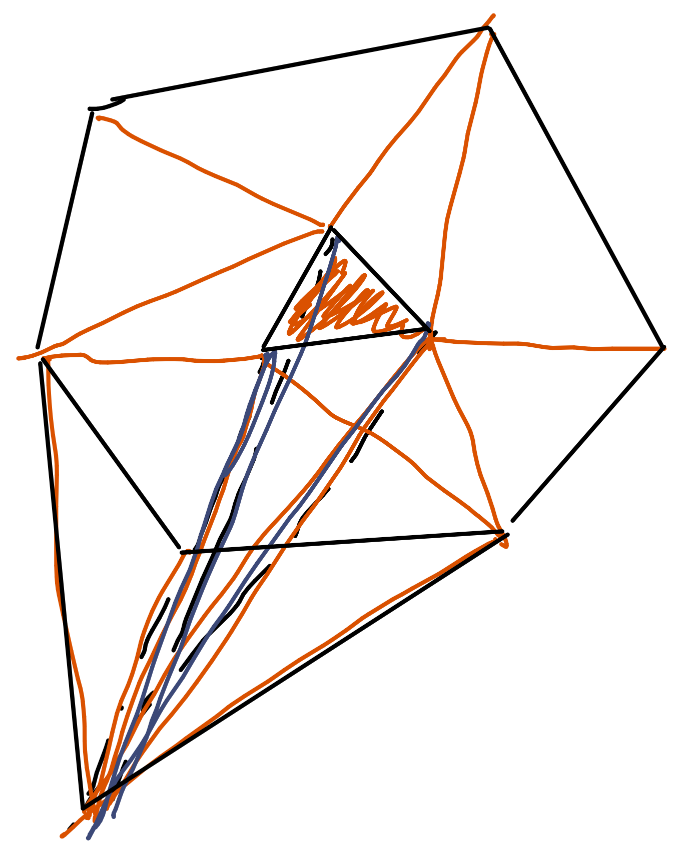
Consider a map f from $\mathbb{R}^n$ into $[0,1]$ defined by sending each ray of C into 1 and the other ones to 0, with respect to some some common triangulation $\mathcal{L}$ of $\mathbb{R}^n$ and C

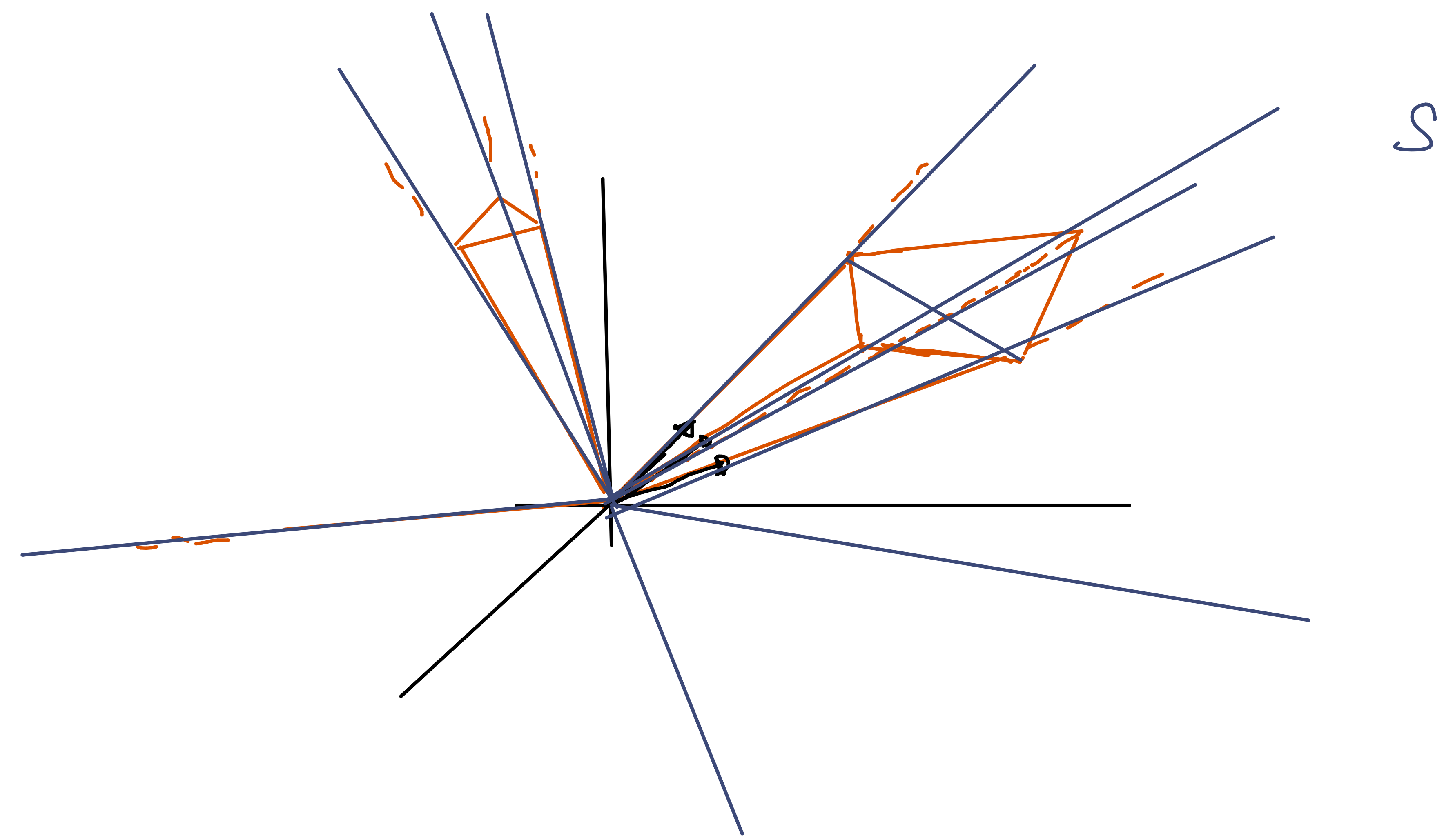
$\mathcal{L} \cap C$ will be called complete if $f^{-1}(1) = C$

3 Every triangulation of $\mathbb{R}^n$ can be refined to a complete triangulation

4 Again by blow up's (on the Farey medients) we transform S' into a unimodular triangulation S''.

5 In order to build a piece-wise homogeneous map from $\mathbb{R}^n$ into C it is enough to give a combinatorial map from S'' to $S'' \cap C$ which sends each ray of S'' in C into itself and all the other rays into 0.





The last topic of the course is Yosida duality.

When we spoke about Hölder theorem we mentioned that any positive $\bar{g} \in G$ induces a map from the archimedean ℓ-group G in $\mathbb{R}$

$$\|g\| := \inf \left\{ \frac{p}{q} \in \mathbb{Q} \;\middle|\; p, q \in \mathbb{N},\ q \neq 0 \quad q|g| \leq p\bar{g} \right\}$$

The function $\| \ \|$ is then a norm which naturally induces a metric $d(g,h) := \|g-h\|$

An ℓ-group is called metrically complete if it is Cauchy-complete with regards to the metric $d(-,-)$

Yosida duality : The category of archimedean, unital, Riesz spaces that are metrically complete is dually equivalent to the category of compact Hausdorff spaces with continuous maps.

Kakutani (M-spaces), Stone - Gelfand (Commutative C^*-algebras) Krein brothers.

The main idea is that there is a more general adjunction between algebras in the language of unital Riesz spaces and the category of topological spaces which is induced by a dualizing object : $\mathbb{R}$

$$\begin{tikzcd}
\mathrm{Alg}_R \arrow[rr, bend left, "\hom_{Alg}(-,\mathbb{R}) = \mathrm{Max}(-)"] & & \mathrm{Top} \arrow[ll, bend left, "\hom_{Top}(-,\mathbb{R})"] \\
& \mathbb{R} \arrow[ul, bend left] \arrow[ur, bend right] &
\end{tikzcd}$$

The general theory of "Concrete Dualities" Porst & Tholen gives an easy to verify condition in order to establish that these two functors form an adjunction.

$$X \xrightarrow{ev} \mathbb{R}^{\hom_{Top}(X,\mathbb{R})}$$

$$x \longmapsto (f(x))_{f \in \hom \dots}$$