Lecture 2 (22/3/24)

Examples
- $(\mathbb{R}, +, -, 0, \min, \max)$
- $(\mathbb{Q}, +, -, 0, \min, \max)$
- $(\mathbb{Z}, +, -, 0, \min, \max)$
- X top. space $\quad C(X) := \{ f : X \to \mathbb{R} \mid f \text{ continuous} \}$

  the operations of $\mathbb{R}$ get pulled back to $C(X)$

  $$f+g\,(x) := f(x)+g(x), \quad f \wedge g\,(x) := f(x) \wedge g(x)$$
- I any set $\quad \mathbb{R}^I := \{ f : I \to \mathbb{R} \mid f \text{ function} \}, \quad \mathbb{Z}^I, \dots$
- $PL_k := \{ f : \mathbb{R}^k \to \mathbb{R} \mid f \text{ is piecewise linear} \}$
- $Z_k := \{ f : \mathbb{R}^k \to \mathbb{R} \mid f \text{ is piecewise linear with integer coefficients} \}$

  e.g. $\underbrace{a_0 + a_1x_1 + a_2x_2 + \dots + a_nx_n}_{\text{linear piece with integer coeff.}} \quad a_i \in \mathbb{Z} \quad 0 \leq i \leq n$

Notice that PL_k can be endowed with a multiplication by a scalar in $\mathbb{R}$, while Z_k cannot.

Notice also that linearity can be replaced by homogeneity.

We defined $|g| := g^+ + g^-$ where $g^+ := g \vee 0$ $\quad g^- := -g \vee 0$

Lemma 2.1 $|g| = g^+ \vee g^-$

Proof Using that $x+y = (x \vee y) + (x \wedge y)$ we have that

$$|g| := g^+ + g^- = (g^+ \vee g^-) + (g^+ \wedge g^-) = g^+ \vee g^-$$

Lemma 2.2 $|g+h| \leq |g| + |h|$

Proof $|g+h| = ((g+h) \vee 0) + ((-g-h) \vee 0) \leq g \vee 0 + h \vee 0 + (-g \vee 0) - (h \vee 0)$

$= g \vee 0 + (-g \vee 0) + h \vee 0 + (-h \vee 0)$

$= |g| + |h|$

Exercise
$(x+y) \vee z \leq (x \vee z) + (y \vee z)$

In general if in a lattice holds $\forall x \forall y \forall z$ $\boxed{x \vee z = y \vee z \text{ and } x \wedge z = y \wedge z \implies x = y}$

then the lattice is distributive.

Remark $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \iff x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Furthermore $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$ is true in any lattice.

Indeed. $x \wedge (y \vee z) \geq x \wedge y$ and $x \wedge (y \vee z) \geq x \wedge z$

$\Uparrow$ $y \leq y \vee z$ $\Uparrow$ $z \leq y \vee z$

Lemma 2.3 In any ℓ-group $\boxed{x \vee z = y \vee z \text{ and } x \wedge z = y \wedge z \implies x = y}$

Proof

$$x = (x \vee z) - z + (x \wedge z) = (y \vee z) - z + (y \wedge z) = y$$

Lemma 2.4 For an abelian group G the following are equivalent

(i) G admits a linear order (compatible with the operations)
(ii) G admits a lattice order
(iii) G is torsion free.

Proof (i) $\Rightarrow$ (ii) obviously. We have seen (ii) $\Rightarrow$ (iii).

To prove (iii) $\Rightarrow$ (i), notice that if G is torsion free then it can be embedded into a divisible group G^d. The latter may be represented as $\prod_{\lambda \in \Lambda} \mathbb{Q}_\lambda$. Use A.C. to well-order Λ

$$G^{d+} := \{ (q_\lambda)_{\lambda \in \Lambda} \mid q_{\min\{\lambda \mid q_\lambda \neq 0\}} > 0 \}$$

It is immediate to verify G^{d+} is closed under $+$ and $G^{d+} \cap -G^{d+} = \emptyset$

$\square$

An ℓ-group homomorphism ($= \ell$-homomorphism) is a function that is both a group homo and a lattice homomorphism.

Def Let H be a subgroup of an ℓ-group G. We say that H is solid (or convex) if $h, k \in H$, $g \in G$, $h < g < k \Rightarrow g \in H$

Def We simply call ℓ-ideal any (normal) solid subgroup of G which is also a sublattice.

Remark: Solidity is equivalent to $h \in H, g \in G \quad |g| < |h| \Rightarrow g \in G$.

Theorem Let $\phi : G \to H$ be a onto ℓ-homo. Then

(1) $\mathrm{Ker}(\phi) := \{ g \in G \mid \phi(g) = 0 \}$ is an ℓ-ideal

(2) If N is an ℓ-ideal then G/N is canonically endowed with the structure of an ℓ-group.

(3) $G/_{\mathrm{Ker}(\phi)} \cong H$

Proof (1) Is straightforward.

(2) We know the G/N is a group. Let us define a lattice order

The coset of g under the equivalence induced by N

$N+g \geq N+h \overset{\text{def}}{\Longleftrightarrow} k+g \geq h$ for some $k \in N$

· First we check that it is well defined.

$N+g = N+g'$, $N+h = N+h'$ $k+g \geq h \Rightarrow k'+g' \geq h'$ for some $k, k' \in N$

$\Downarrow$ $\Downarrow$

$\underbrace{g'-g}_{m_1} \in N$ $\underbrace{h'-h}_{m_2} \in N$ $g = g' - m_1$ $h = h' - m_2$

$k+g \geq h \Rightarrow k+g'-m_1 \geq h'-m_2 \Rightarrow \underbrace{(k-m_1+m_2)}_{\in N} + g' \geq h'$

· Check antisymmetry; assume $N+g \geq N+h$ and $N+h \geq N+g$

By def. $\exists\, m, n \in N$ s.t. $g+m \geq h$ and $h+n \geq g$

$\Rightarrow h+m+n \geq g+m \geq h \Rightarrow m+n \geq g-h+m \geq 0$; $m+n \in N$

$\overset{\text{solidity}}{\Longrightarrow} g-h+m \in N \Rightarrow g-h \in N$ therefore $N+g = N+h$

Transitivity is tedious but easy.

Translation invariance is straightforward because $N+g \leq N+h$ implies $\exists\, m \in N$ $m+g \leq h$. Hence for any k $N+g+k \leq N+h+k$.

We now check that $\boxed{N+g\vee h = (N+g)\vee(N+h)}$ $\left(\pi_N(g\vee h) = \pi_N(g)\vee\pi_N(h)\right)$

Clearly, since $g\vee h \geq g, h$ we get $N+g\vee h \geq (N+g)\vee(N+h)$.

Now suppose $d\in G$ and $N+d \geq N+g, N+h \Rightarrow \exists\, m, n\in N$ s.t.

$m+d \geq g$ and $n+d\geq h \Rightarrow \underset{\substack{\Cap\\ N}}{(m\vee n)}+d \geq g\vee h$

$\Rightarrow N+d \geq N+g\vee h$

$\Rightarrow (N+g)\vee(N+h) \geq N+(g\vee h)$. This shows that G/N is a join-semilattice. Hence G/N is a ℓ-group $\square$