Lecture 3 26/3/24

Lemma 3.1 The lattice reduct of any ℓ-group is distributive.

<u>Proof</u> It suffices to prove $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Since $x \vee (y \wedge z) \leq x \vee y$ and $x \vee (y \wedge z) \leq x \vee z$

we obtain $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$.

For the other inequality set $t = y \wedge z$. Then we have

$0 \leq y - (y \wedge z) = y - t \quad \Rightarrow \quad x \leq y - t + x \leq y - t + (t \vee x)$

$y - (y \wedge z) = y + (-y \vee -z) = 0 \vee y - z$

Since $0 \leq (t \vee x) - t$ we obtain $y \leq y - t + (t \vee x)$

So we can conclude $x \vee y \leq y - t + (t \vee x)$ and similarly
$x \vee z \leq z - t + (t \vee x)$

Therefore, $(x \vee y) \wedge (x \vee z) \leq (y - t + (t \vee x)) \wedge (z - t + (t \vee x)) =$

$= (y \wedge z) - t + (t \vee x) = t \vee x = x \vee (y \wedge x)$ $\square$

Lemma 3.2 If in an ℓ-group $|x+y| \le |x| + |y|$ holds then the ℓ-group is Abelian.

<u>Proof</u>

It suffices to prove that $+$ is commutative among elements of the positive cone.

Let $x, y \in G^+$

$$x+y = |x+y| \underset{= |-(x+y)|}{\le} |-y-x| \le |-y| + |-x| = |y|+|x| = y+x$$

By interchanging x with y we also obtain

$$y+x \le x+y$$

$\square$

Lemma 3.3 If G is an ℓ-group if H is a convex subgroup of G the H is also a sublattice

Proof

Preliminary we prove that $x \vee y = (x-y)^+ + y$

$$x \vee y = x - (x \wedge y) + y = x + (-x \vee -y) + y = 0 \vee (x-y) + y = (x-y)^+$$

Also observe that $x \wedge y = -(-x \vee -y)$

Remember that a subgroup H is convex iff $\forall x \in G \ \forall h \in H$

$0 \leq x \leq |h| \implies x \in H$. In particular this implies that

$h \in H$ then $h^+ \in H$ because $0 \leq h^+ \leq |h|$

So to prove that H is a sublattice, consider $x, y \in H$

then $x - y \in H \implies (x-y^+) \in H \implies x \vee y = (x-y)^+ + y \in H$

$\square$

Lemma 3.4. Let $f: G \to H$ be a group homomorphism between ℓ-groups.
TFAE:

(a) f is an ℓ-homomorphism ($=$ group $+$ lattice homomorp.)

(b) $f(x^+) = \big(f(x)\big)^+$

(c) $x \wedge y = 0$ (x and y are orthogonal) $\Rightarrow$ $f(x) \wedge f(y) = 0$

(d) $f(|x|) = |f(x)|$.

<u>Proof</u> $a \Rightarrow b$, $a \Rightarrow c$ and $a \Rightarrow d$ are clear.

$(c) \Rightarrow (b)$ Observe that $a \wedge b = 0$ then $a = (a-b)^+$. Indeed, $-a \vee -b = 0$
so $a = a + 0 = a + (-a \vee -b) = 0 \vee a - b = (a-b)^+$
Recall that $x^+ \wedge x^- = 0$, therefore $f(x^+) \wedge f(x^-) = 0$

$$f(x^+) = \big(f(x^+) - f(x^-)\big)^+ \overset{\text{group homo}}{=} \big(f(x^+ - x^-)\big)^+ = \big(f(x)\big)^+$$

$(b) \Rightarrow (a)$
$$f(x \vee y) = f\big((x-y)^+ + y\big) = f\big((x-y)^+\big) + f(y) \overset{(b)}{=} \big(f(x-y)\big)^+ + f(y)$$
$$= \big(f(x) - f(y)\big)^+ + f(y) = f(x) \vee f(y)$$

(d) $\Rightarrow$ (b) Preliminarly observe

$$2x^+ := x^+ + x^+ = (x \vee 0) + (x \vee 0) = ((x \vee 0) + x) \vee (x \vee 0 + 0)$$
$$= (\underline{(x \vee 0)} + x) \vee (x + (-x \vee 0)) =$$
$$= x + ((x \vee 0) \vee (-x \vee 0)) = x + |x|$$

$$2f(x^+) = f(2x^+) = f(x + |x|) = f(x) + f(|x|) = f(x) + |f(x)|$$
$$f(x) + |f(x)| = 2(f(x))^+$$

Hence $2f(x^+) = 2(f(x))^+$ by torsion-freeness

$$\Downarrow$$

$$0 = 2f(x^+) - 2(f(x))^+ = 2(f(x^+) - (f(x))^+)$$

$$\Downarrow$$

$$f(x^+) - (f(x))^+ = 0 \Rightarrow f(x^+) = (f(x))^+$$

$\square$

If $X \subseteq G$ we write $C(X)$ for the smallest ℓ-ideal containing X

Lemma 3.5 $C(X) = \overset{D}{\overset{\shortparallel}{\{ g \in G \mid |g| \leq |x_1| + \dots + |x_n| \text{ for some } x_1 \dots x_n \in X \}}}$

<u>Proof</u> It is clear that D must be contained in all ℓ-ideals containing X. Therefore, it suffices to prove that D is an ℓ-ideal. We only need to prove that D is a subgroup, because it is obvious that it is convex.

Let $g, h \in D \Rightarrow \exists\ x_1 \dots x_n, y_1, \dots, y_m \in X$ s.t

$|g| \leq |x_1| + \dots + |x_n|$ and $|h| \leq |y_1| + \dots + |y_m|$ so

$|g+h| \leq |g| + |h| \leq |x_1| + \dots + |x_n| + |y_1| + \dots + |y_m|$

Hence $g+h \in D$. $\square$

Observe that the subgroup generated by a family of ℓ-ideals is an ℓ-ideal

Lemma 3.6 The lattice of ℓ-ideals is a complete Heyting algebra.

Proof It is clear that ℓ-ideals form a complete lattice because they are closed under arbitrary intersections. Thus the join of an infinite family is the intersection of all ℓ-ideals containing that family.

To prove that it is a Heyting algebra it is enough to check

$$A \cap \left(\bigvee_i B_i \right) = \bigvee_i (A \cap B_i)$$

One inclusion should obvious $\bigvee_i (A \cap B_i) \subseteq A \cap \left(\bigvee_i B_i \right)$.

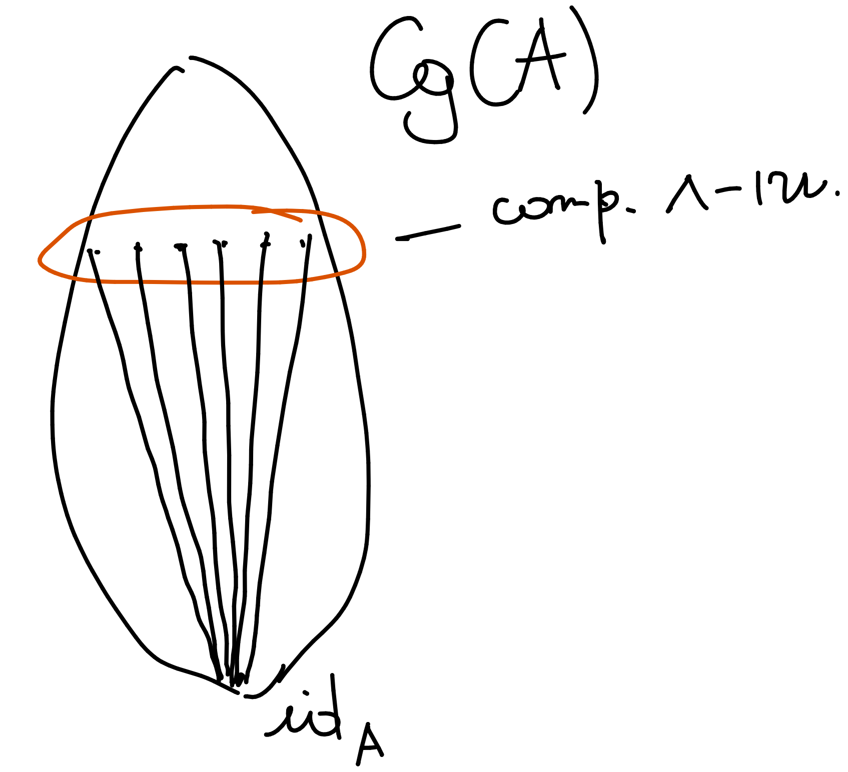
For the other take $0 \le x \in A \cap (\bigvee B_i) \Rightarrow x = b_1 + \dots + b_n$ for some b_j with $1 \le j \le n$ belonging to some B_i . Furthermore each $b_j \in A$ because $0 \le b_j \le x$ so $x \in \bigvee (A \cap B_i)$

$\square$

By a result of Birkhoff every algebra is the subdirect product of subdirectly irreducible algebras. Also, the subdirectly irreducible factors correspond to the completely meet irreducible congruences of the algebra

Def An element b of a lattice L is called *completely meet irreducible* if $b = \bigwedge X$ with $X \subseteq L \implies b \in X$

An element is called *meet irreducible* if the condition above holds for finite X's.



Theorem Let P be an ideal of a lattice ordered group. TFAE

(a) P is prime (G/P is linearly ordered)

(b) $\{C \mid C$ ℓ-ideal $C \supseteq P\}$ is linearly ordered

(c) P is meet-irreducible

(d) If $a \wedge b \in P$ then $a \in P$ or $b \in P$

(e) If $a \wedge b = 0$ then $a \in P$ or $b \in P$.

Proof (a) $\Rightarrow$ (b) Take B and C ℓ-ideals that extend P and are incomparable $b \in B^+ \setminus C$ $c \in C^+ \setminus B$. W.l.o.g assume

$P+b \geq P+c \Rightarrow \exists p \;\; p+b \geq c \Rightarrow$ notice the $p+b \in B$

$\Rightarrow c \in B$ $\lightning$

(b) $\Rightarrow$ (c) If $B \cap C = P \Rightarrow B, C \supseteq P$ w.l.o.g. $B \supseteq C$ but then

$P = C$

(c) $\Rightarrow$ (d) Take $a \wedge b \in P$ $\qquad a\wedge b \in C(a), C(b)$

$$(P \vee C(a)) \wedge (P \vee C(b)) \overset{\text{distributivity}}{=} P \vee (C(a) \wedge C(b)) \overset{\supseteq}{=} P \vee (C(a\wedge b)) = P$$

$\Rightarrow$ either $P \vee C(a)) = P$ or $P \vee C(b) = P$

$\Rightarrow$ either $a \in P$ or $b \in P$

(d) $\Rightarrow$ (e) Obvious

(e) $\Rightarrow$ (a) $g, h \in G$ we want to show that $\begin{matrix}(g \vee h) - g \\ \text{or} \\ (g \vee h) - h\end{matrix} \in P$

Because this would give us $P + h = P + h = P + (g \vee h) \geq P + g$

To prove the claim

$$((g \vee h) - g) \wedge ((g \vee h) - h) = (g \vee h) + (-g \wedge -h) = (g \vee h) - (g \vee h) = 0$$

$\square$

Corollary Any abelian ℓ-group is a subdirect product of linearly ordered ℓ-groups.