Lecture 4 27/3/24

Let τ be a finitary language (all operations have finite arity)

Theorem (Birkhoff)
Every τ-algebra A is a subdirect product of subdirectly irreducible τ-algebras. In particular $A \underset{s}{\hookrightarrow} \prod_{\theta} A/_{\theta}$ with θ comp $\wedge$-irreducible congruence of A.

Def An ℓ-group G is called archimedean if $\forall x, y \in G$
$$(\forall n \in \mathbb{N} \quad nx \leq y) \Longrightarrow x \leq 0$$

Intuitively one can think of archimedean ℓ-groups as ℓ-groups of functions with values in $\mathbb{R}$.

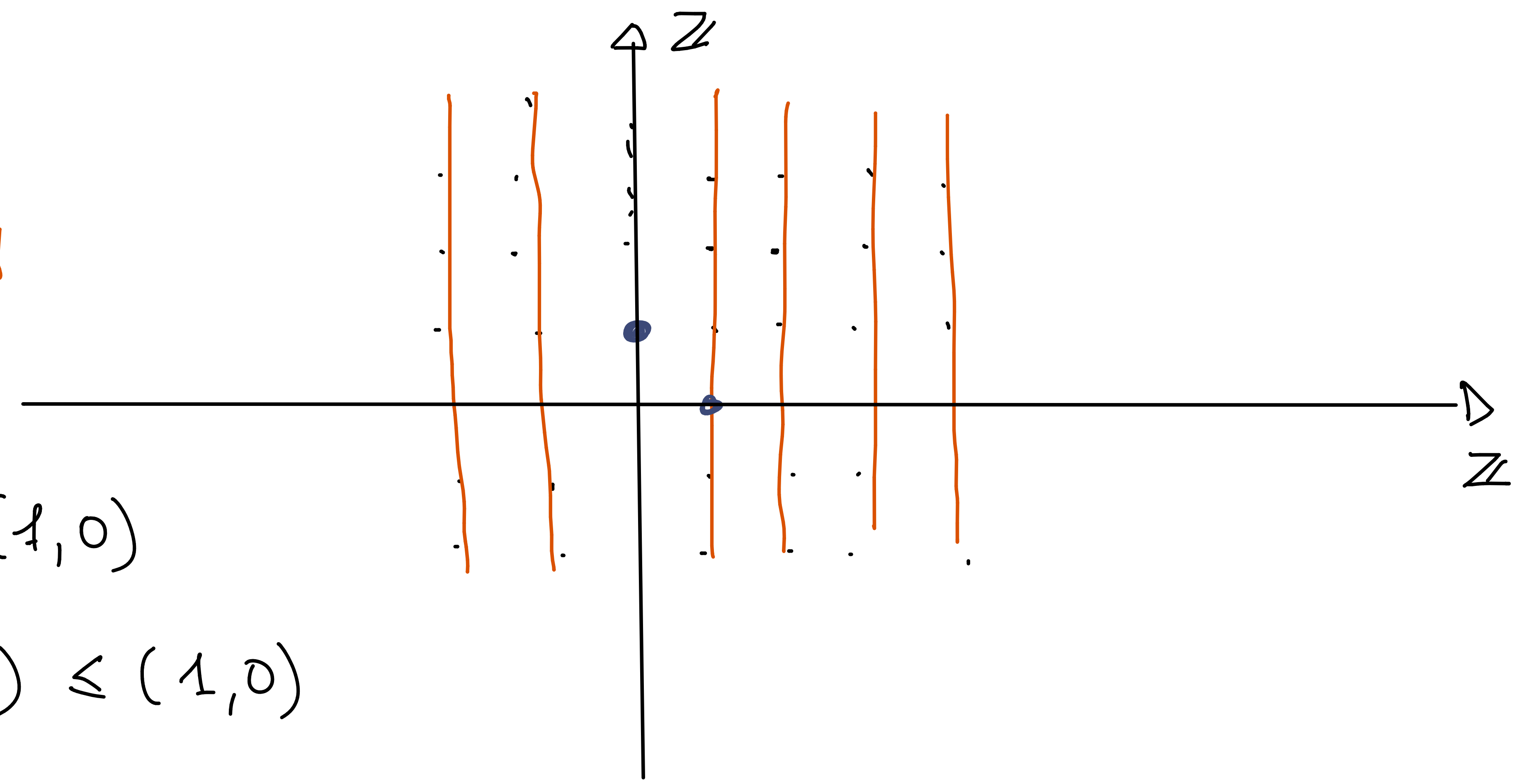
Example If X is a compact Hausdorff space then $C(X) = \{ f : X \to \mathbb{R} \mid f \text{ contin} \}$ is archimedean: $f, g \in C(X)$ suppose $\forall n \in \mathbb{N}$ $nf \leq g$
this means that $\forall x$ $nf(x) \leq g(x) \Rightarrow f(x) \leq 0$

Def Let G and H be two linearly ordered ℓ-groups, we define $G \vec{\times} H = G \times_{lex} H$ to be the direct product of the groups underlying G and H endowed with lexicographic order

$(g,h) < (g',h')$ if $g < g'$ or if $h < h'$ whenever $g = g'$

$\mathbb{Z} \vec{\times} \mathbb{Z}$

is linearly ordered



Take $(0,1)$ and $(1,0)$

$\forall\, n \in \mathbb{N} \quad n(0,1) = (0,n) \leq (1,0)$

yet $(0,1) \not\leq (0,0)$

Berman theorem Every archimedean ℓ-group is abelian

Hölder theorem Every linearly ordered archimedean ℓ-group embeds into $\mathbb{R}$

Hahn theorem Every linearly ordered abelian ℓ-group embeds into a lexicographic power of $\mathbb{R}$.

Hölder theorem Let G be a linearly ordered group. T.F.A.E

1) G is archimedean

2) G ℓ-embeds into $\mathbb{R}$

3) G has exactly two ℓ-ideals. $\{0\}$, G

Prof 1) $\Rightarrow$ 2) Choose $g \in G^+$ and define $\phi : G \to \mathbb{R}$ as follows

$$L(h) := \left\{ \frac{m}{n} \in \mathbb{Q} \;\middle|\; mg > nh \text{ with } \underset{n \neq 0}{m, n \in \mathbb{Z}} \right\}$$

$$mg := \overbrace{g + g + \dots + g}^{m \text{ times}}$$

$L(h)$ is non empty because G is archimedean and linearly ordered : $\exists m \in \mathbb{N}$ $mg > h$

$L(h)$ is bounded below $\exists n \in \mathbb{N}$ $n|h| > g$

Define $\quad \phi(h) = \bigwedge L(h)$ the $\bigwedge$ is taken in $\mathbb{R}$

Let us show that ϕ is a ℓ-homomorphism.

Let $h, k \in G$ let $m, n, p, q \in \mathbb{Z}$ we are going to show that

(A) $\quad \overset{\in L(h)}{\dfrac{m}{n}} \geq \phi(h)$ and $\overset{\in L(k)}{\dfrac{p}{q}} \geq \phi(k)$ then $\dfrac{m}{n} + \dfrac{p}{q} \geq \phi(h+k)$

so that we have $\phi(h) + \phi(k) \geq \phi(h+k)$

To prove (A) assume $mg > nh$ and $pg > qk$

$(mq + np)\, g = mqg + npg > nqh + nqk = nq(h+k) \Longrightarrow \boxed{(mq+np)\, g > nq(h+k)}$

Thus $\quad \dfrac{m}{n} + \dfrac{p}{q} = \dfrac{mq + np}{nq} \geq \phi(h+k) \Longrightarrow \phi(h) + \phi(k) \geq \phi(h+k)$

Similarly one can prove that if $\frac{m}{n} \leq \phi(h)$ and $\frac{p}{q} \leq \phi(k)$

then $\frac{m}{n} + \frac{p}{q} \leq \phi(h+k) \qquad \Longrightarrow \quad \phi(h) + \phi(k) \leq \phi(h+k)$

All together $\quad \phi(h) + \phi(k) = \phi(h+k)$

ϕ is clearly order preserving. Because if $h_1 \leq h_2$

then $L(h_1) \supseteq L(h_2)$ so $\bigwedge L(h_1) \leq \bigwedge L(h_2)$.

To see that ϕ is an embedding let $h>0$ then $\exists n \in \mathbb{Z}$ $ng > h$ by archimedeanity so

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$(2) \Rightarrow (3)$ To prove (3) it is enough to show the $\mathbb{R}$ has exactly two ideals. To this end, it is enough to notice that for any nonzero $r \in \mathbb{R}$ the ℓ-ideal generated by r, $C(r)$ is the whole $\mathbb{R}$.

Recall

$$C(r) := \{ x \in \mathbb{R} \mid |x| < n|r| \text{ for some } n \in \mathbb{N} \} = \mathbb{R}$$

$(3) \Rightarrow (1)$ For any $g \in G$ $g \neq 0$ $C(g) = G$ so

$$C(g) = \{ x \in G \mid |x| < n|g| \} = G$$

This says that for every $x, y \in G$ $\exists n \in \mathbb{N}$ s.t

$$n|y| > |x|$$

and this is equivalent to archimedeanity.

$\square$

Lemma (Elliot 1979). Let G be a linearly ordered ℓ-group.
Let $P := \{p_1 \dots p_m\} \subseteq G^+$. Then there exist $C \subseteq G^+$

(1) C is a basis for the subgroup of G generated by P

(2) C is obtained from P by a finite number of "subtractions"

$$\boxed{P' = \left(P \setminus \{p_i\}\right) \cup \{p_i - p_j\}}$$

Weinberg Theorem: An equation is true in all abelian ℓ-groups iff it is true in $\mathbb{Z}$.

Proof Consider an equation $w(x_1, \dots, x_n) = 0$ which does not hold in all abelian ℓ-groups. Let G be an ℓ-group and $g_1 \dots, g_n \in G$ s.t. $w(g_1 \dots g_n) \neq 0_G$. Since G is a subdirect product of ℓ.o ℓ-groups we assume that G is linearly ordered.

$P_1 := \{ p_i \mid 0 \le i \le n ,\ p_i > 0 \}$

$$p_i = \begin{cases} w_i(g_1 \dots g_n) & w_i(g_1 \dots g_n) \ge 0 \\ -w_i(g_1 \dots g_n) & w_i(g_1 \dots g_n) < 0 \end{cases}$$

where w_i are all subterms of w with $w_0 = w$

$P_2 := \{ p_i - p_j \mid p_i, p_j \in P_1$ and $p_i - p_j > 0 \}$

Apply Elliott lemma to $P := P_1 \cup P_2$ so to obtain a subset $C \subseteq G^+ \setminus \{0\}$ that is a basis for the group generated by P. Since we are in abelian groups and C is a basis the group generated by P must be isomorphic to $\mathbb{Z}^{|C|}$.

Set $r := |C|$, endow $\mathbb{Z}^r$ with the componentwise order, for $(x_1 .. x_r) \in \mathbb{Z}^r$, $(x_1 .. x_r) \succcurlyeq 0$ iff $\forall i \le r$ $x_i \succcurlyeq 0$.

CLAIM: $p_i > p_j$ in G iff $p_i \succ p_j$ in $\mathbb{Z}^r$

If $p_i > p_j$ then $p_i - p_j \in P_2$ and P_2 belongs to the positive span of C

Then $p_i - p_j \succ 0 \Rightarrow p_i \succ p_j$. Vice versa is similar

So

$$w_i(g_1 \dots g_n) > 0 \text{ in } G \text{ iff } w_i(g_1 \dots g_n) \succ 0 \text{ in } \mathbb{Z}^2$$

So in particular $w = w_0 > 0$ in G iff $w_0 \succ 0$ in $\mathbb{Z}^2$

Therefore $w(g_1 .. g_n) \neq 0$ in $\mathbb{Z}^2$ so the equation

$$w(x_1 .. x_n) = 0 \text{ must fail in } \mathbb{Z}.$$

$\square$

V. Marra Free lattice-ordered abelian groups. Forum Mathematicum.

$$\|v - w\| := \bigwedge \left\{ \frac{p}{q} \in \mathbb{Q} \;\middle|\; p \geq 0,\ q > 0 \quad p \geq q\,|v - w| \right\}$$