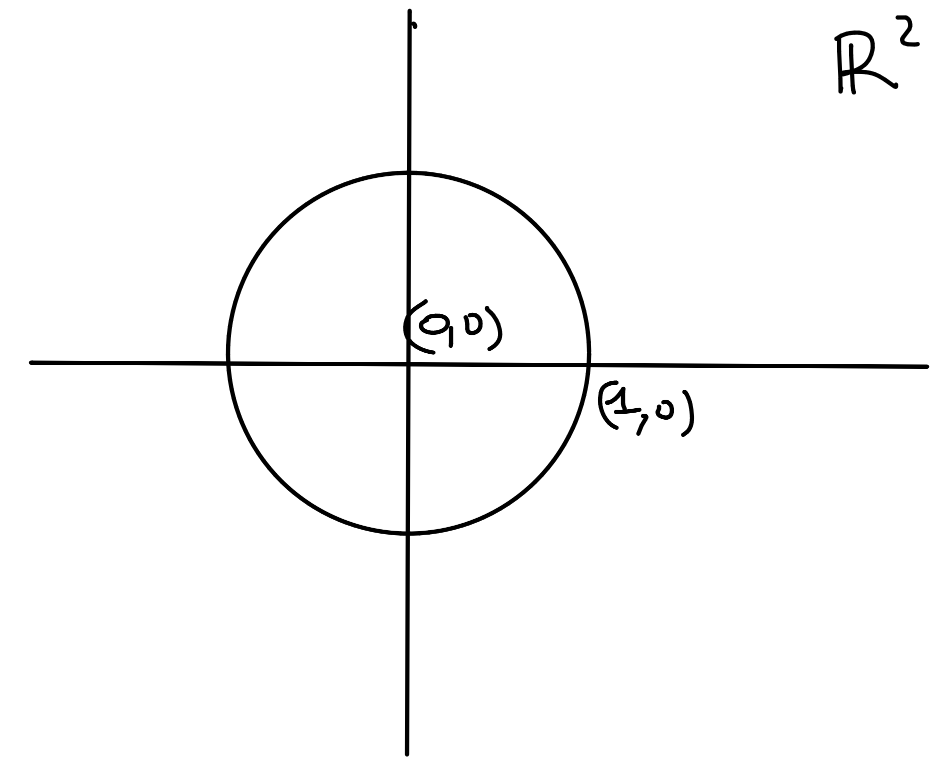
Lecture 5 4|4|24

$$x^2 + y^2 - 1 = 0$$

$$2x^2 + 2y^2 - 2 = 0$$



In general there are two operators : $\mathbb{V}$, $\mathbb{I}$

$$\begin{tikzcd} \mathcal{P}(K[x_1 \ldots x_n]) \arrow[r, bend left, "\mathbb{V}"] & \mathcal{P}(K^n) \arrow[l, bend left, "\mathbb{I}"] \end{tikzcd}$$

$R \subseteq K[x_1 \ldots x_n]$ $\qquad$ $\mathbb{V}(R) := \{ s \in K^n \mid p(s) = 0 \quad \forall p \in R \}$

$S \subseteq K^n$ $\qquad$ $\mathbb{I}(S) := \{ p \in K[x_1 \ldots x_n] \mid p(s) = 0 \quad \forall s \in S \}$

Let V be a variety of algebras

Let $A \in V$

By Birkhoff result V has free objects $\mathcal{F}_V(\kappa)$ for any cardinal κ

Fix κ a cardinal

$R \subseteq \mathcal{F}_V(\kappa)^2$ $\qquad \mathbb{V}(R) := \{ s \in A^\kappa \mid A \models p(s) = q(s) \ \forall (p,q) \in R \}$

$S \subseteq A^\kappa$ $\qquad \mathbb{I}(S) := \{ (p,q) \in \mathcal{F}_V(\kappa)^2 \mid A \models p(s) = q(s) \ \forall s \in S \}$

<u>Lemma</u> $\mathbb{V}$ and $\mathbb{I}$ form a (contravariant) Galois connection

We now try to lift the Galois connection to an adjunction

Notice that $\mathbb{I}(S)$ is always a congruence of $\mathcal{F}_V(\kappa)$

Thus, a first category to be considered is V_p = class of all "presented" algebras in V

$V_p \simeq V$ $\qquad \mathcal{F}_V(\lambda)/\theta$ for some λ, θ

A second category is given by all possible subsets of A^k for k ranging among cardinals. Which arrows ?

Def Let λ be a cardinal and $f: A^\lambda \to A$ a function. We say that f is definable if $\exists\, t \in \mathcal{F}_V(\lambda)$ s.t $\forall\, a \in A^\lambda$

$$f(a) = t(a)$$

This generalises to $f: A^\lambda \to A^k$, it is definable if there exist $(t_\alpha)_{\alpha < k}$ s.t. $\forall\, a \in A^\lambda$

$$f(a) = (t_\alpha(a))_{\alpha < k}$$

This can be further generalised to $f: S \subseteq A^\lambda \to T \subseteq A^k$, it is definable if $\exists\, f': A^\lambda \to A^k$ definable s.t. $\forall\, s \in S$

$$f(s) = f'(s)$$

So we have two categories

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "\mathcal{V}"] & \mathcal{D} \arrow[l, bend left, "\mathcal{I}"] \end{tikzcd}$$

The functor $\mathcal{I} : \mathcal{D} \to \mathcal{V}$

on object $S \subseteq A^k$ $\quad \mathcal{I}(S) := \mathcal{F}_V(k) / \mathbb{I}(S)$

on arrows $\lambda : S \subseteq A^u \to T \subseteq A^v$ definable thus $\exists (\ell_\beta)_{\beta<v}$ $\lambda(a) = (\ell_\beta(a))_{\beta<v}$

$$\mathcal{I}(\lambda) : \mathcal{I}(T) \to \mathcal{I}(S)$$

$$\frac{s((x_\beta)_{\beta<v})}{\mathbb{I}(T)} \in \mathcal{I}(T) \longmapsto \frac{s(x_\beta/\ell_\beta)_{\beta<v}}{\mathbb{I}(S)}$$

where $(\ell_\beta)_{\beta<v}$ are the defining terms of λ

The functor $\mathcal{V}: V \to D$

on objects $\mathcal{V}\left(\mathcal{F}_\nu(\kappa)/\theta\right) := \mathbb{V}(\theta) \subseteq A^\kappa$

on arrows $h: \mathcal{F}_\mu/\theta_1 \longrightarrow \mathcal{F}_\nu/\theta_2$ homomorphism

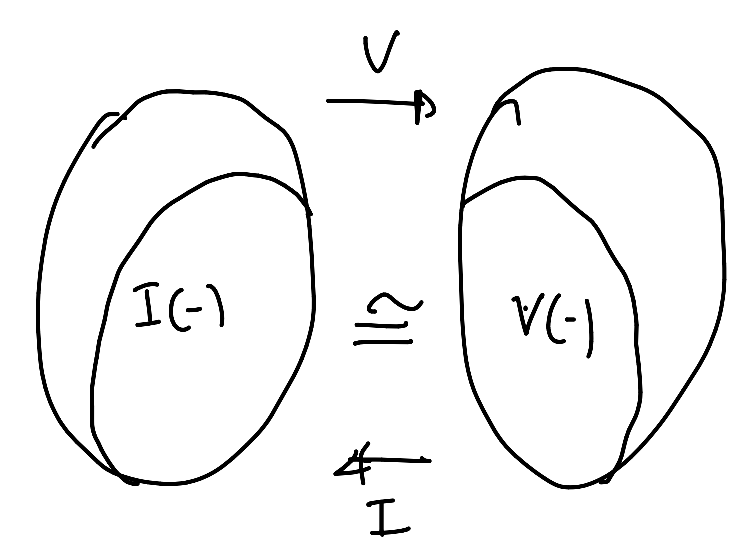
$$f_\alpha \in h\left(\frac{(x_\alpha)}{\theta_1}\right)$$

$$\mathcal{V}(h): \mathbb{V}(\theta_2) \longrightarrow \mathbb{V}(\theta_1)$$

Let $(s_\beta)_{\beta<\nu} \in \mathbb{V}(\theta_2)$ $\quad \mathcal{V}(h)(s_\beta)_{\beta<\nu} := \left(f_\alpha\left((s_\beta)_{\beta<\nu}\right)\right)_{\alpha<\mu} \in \mathbb{V}(\theta_1)$

Lemma The functor $\mathcal{V}$ and $\mathcal{I}$ form a contravariant adjunction

$$\mathcal{V} \dashv \mathcal{I}$$



We now turn our attention to the fixed

$S \subseteq A^k$ $\quad \mathcal{V} \circ \mathcal{Y}(S) = \mathcal{V}\left(F_v(k)/_{\mathbb{I}(S)} \right) = \mathbb{V}\mathbb{I}(S)$

$F_v(k)/_{\theta}$ $\quad \mathcal{Y} \circ \mathcal{V}\left(F_v(k)/_{\theta} \right) = \mathcal{Y}(\mathbb{V}(\theta)) = F_v(k)/_{\mathbb{I}\mathbb{V}(\theta)}$

Fixed points in D $\qquad S = \mathbb{V}\mathbb{I}(S)$

Fixed point in V $\qquad F_v(k)/_{\theta} \cong F_v(k)/_{\mathbb{I}\mathbb{V}(\theta)}$

Lemma TFAE

1) $\theta = \mathbb{I}\mathbb{V}(\theta)$

2) $\theta = \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{I}(\{a\})$

3) $F_v(k)/_{\theta} \longrightarrow \prod_{a \in \mathbb{V}(\theta)} F_v(k)/_{\mathbb{I}(\{a\})}$

Conseq. of universal algebra

$a \in k^n$

$\mathbb{I}(a) = \{ p \in k[x_1 \ldots x_n] \mid p(a) = 0 \}$

is a maximal ideal

Compare the previous lemma with Hilbert Nullstellensatz

An ideal I is s.t. $I = \mathbb{I}\mathbb{V}(I)$ if and only if $I = \bigcap_{\substack{I \subseteq M \\ M \text{ maximal ideal.}}} M$

$\forall a \in \mathbb{V}(\Theta)$

$$\begin{tikzcd} & F(\nu) \arrow[dl, "\hat{a}"'] \arrow[d, "q_a"] \\ A & \frac{F(\nu)}{\mathbb{I}(a)} \arrow[l, dashed, "!"'] \end{tikzcd}$$

Lemma TFAE

1) $\Theta = \mathbb{I}(a)$ for some $a \in A^{\nu}$

2) $\frac{F(\nu)}{\Theta} \longrightarrow A$ is an embedding

$a \in \mathbb{V}(\Theta)$

$\boxed{\mathbb{I}(a) \supseteq \Theta}$ well defin.

$\boxed{\mathbb{I}(a) \subseteq \Theta}$ embedding

Fix $a \in \mathbb{V}(\Theta)$ $\quad \frac{t}{\Theta} \xmapsto{\hat{a}} t(a)$

Let $V = BA$ Boolean algebras

Let $A = 2$

We have the adjunction between BA and subsets of 2^k

k and definable maps.

$\mathcal{F}_V(k)/\theta$ is fixed by the adjunction

if $\theta = \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{I}(a)$. $\Longleftrightarrow$ $\mathcal{F}_V(k)/\theta \hookrightarrow \prod_{a \in \mathbb{V}(\theta)} \mathcal{F}_V(k)/\mathbb{I}(a) = \prod_i 2$

$\theta = \mathbb{I}(a)$ iff $\mathcal{F}_V(k)/\theta \cong 2$

Thus, all BA is fixed in the adjunction

$S \subseteq 2^k$ $\quad S = \mathbb{V}(\mathbb{I}(S)) = \mathbb{V}(\{ f \in \mathcal{F}_k(V) \mid f(s) = 0 \ \forall s \in S \})$

$= \{ s \in 2^k \mid f(s) = 0 \ \forall f \in \mathbb{I}(S) \}$