Lecture 6 5/4/24

Let V be any variety, $A \in V$ any algebra

$$\begin{tikzcd} V \arrow[rr, bend left, "\mathcal{V}"] & & D \arrow[ll, bend left, "\mathcal{I}"] \end{tikzcd}$$

objects: subsets of A^k for $k \in$ Card

arrows: definable maps

$$\mathcal{V}\left(\mathcal{F}_V(K)/\theta \right) := \mathbb{V}(\theta) \qquad \mathcal{I}(S \subseteq A^k) := \mathcal{F}_V(k)/\mathbb{I}(S)$$

Theorem $\mathcal{V} \dashv \mathcal{I}$

Lemma 1 $\mathbb{I}\mathbb{V}(\theta) = \theta$ iff $\theta = \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{I}(a)$ iff $\mathcal{F}_V(k)/\theta \hookrightarrow \prod_{a \in \mathbb{V}(\theta)} \mathcal{F}_V(k)/\mathbb{I}(a)$

Lemma 2 $\theta = \mathbb{I}(a)$ iff $\mathcal{F}_V(k)/\theta \hookrightarrow A$

Example : Set $\mathbb{V}$ = variety of Boolean algebras

$$A = (\{0,1\}, \min, \max, 1-x, 0, 1) =: \mathbf{2}$$

- By Lemma 1 a Boolean algebra B is fixed in the adjunction iff $B \cong F_V(k)/\theta$ and $B \hookrightarrow \prod_{a \in \mathbb{V}(\theta)} F_V(k)/\mathbb{I}(a)$
- By Lemma 2 $F_V(k)/\mathbb{I}(a) \cong \mathbf{2}$

Since every Boolean algebra is a subdirect product of copies of $\mathbf{2}$ (because $\mathbf{2}$ is the only subdirectly irr Boolean algebra) every Boolean algebra is fixed in the adjunction

So we need to characterize the fixed points of $\mathbb{V} \circ \mathbb{I}$.

$S \subseteq A^k$ is fixed iff $\exists\, \theta$ s.t. $S = \mathbb{V}(\theta)$

$$\left(\begin{array}{l} \text{if } S = \mathbb{V}\mathbb{I}(S), \text{ take } \theta = \mathbb{I}(S) \\ \text{if } S = \mathbb{I}(\theta) \Rightarrow \mathbb{V}\mathbb{I}(S) = \mathbb{V}\mathbb{I}\mathbb{V}(\theta) = \mathbb{V}(\theta) = S \end{array} \right)$$

Need to characterise subsets S of the form $\mathbb{V}(\Theta)$

$$\mathbb{V}(\Theta) := \{ s \in A^k \mid p(s) = q(s) \ \ \forall (p,q) \in \Theta \}$$

$$\mathbb{V}(\Theta) = \bigcap_{(p,q)\in\Theta} \mathbb{V}(p,q)$$

$$\mathbb{V}(p,q) := \{ s \in A^k \mid p(s) = q(s) \}$$

Zariski basic closed sets

$$\mathbb{V}(t) := \{ s \in A^k \mid t(s) = 0 \}$$

Stone topology

in Boolean algebras this is equivalent to

$$V(a) := \{ m \in \mathrm{Max}\, B \mid m \ni a \}$$

$$\mathbb{I}(s) \subseteq F_V(k)$$

$$\{ t \mid t(s) = 0 \}$$ this is a maximal ideal

Summing up to characterise the fixed points of $\mathbb{V} \circ \mathbb{I}$ we need to characterise the Zariski closed subsets of A^k (2^k)

Theorem: A topological space is Stone iff it is homeomorphic to a closed subspace of 2^{k}.

Furthermore by functional completeness of $(\wedge, \neg)$ and the fact that $S \subseteq 2^{k} \to T \subseteq 2^{\mu}$ only involves finitely many variables one obtain that definable maps are exactly the continuous maps

Therefore, the category of fixed points in D is equivalent to the category of Stone spaces and continuous maps.

Exercise: Derive Priestley duality for distributive lattices

As you have seen in the course by Serafina Lapenta there is a class of ℓ-groups of special interest: unital ℓ-groups.

Def An element u in an ℓ-group G is called strong order unit if $\forall g \in G \ \exists n \in \mathbb{N} \ \ nu > g$

It is clear (by using a standard compactness argument) that strong order units are not first order definable.
Yet, Mundici's functor establish an equivalence of categories between unital ℓ-groups (= ℓ-groups with a strong order unit) and unit preserving ℓ-homomorphisms and a variety of algebras called MV-algebras.

$$(A, \oplus, \neg, 0)$$

Ex: $([0,1], \oplus, \neg, 0)$ where $x \oplus y := \min(x+y, 1)$
$\neg x := 1 - x$

McNaughton's theorem The free MV-algebra over k generators is isomorphic to the algebra of functions $[0,1]^k \to [0,1]$ which are continuous, piecewise (affine) linear, where each piece has integer coefficients.

Theorem (Chang) Every MV-algebra is a subdirect product of linearly ordered ones.

Hölder's theorem for MV-algebras An MV algebra is simple if and only if it has a (unique) embedding into $[0,1]$

Recall that a semi-simple algebra is a subdirect product of simple algebras. Therefore, semi-simple MV-algebras are exactly the subdirect products of copies of subalgebras of $[0,1]$

Let us apply the duality framework above to MV-algebras.

So let us set $V := MV$, $A := [0,1]$

What are the fixed points on the algebraic side

They are the semisimple MV-algebras.

What are the fixed points on the geometric side

$s \to t := \neg s \oplus t$

$|s-t| := \neg(s \to t) \wedge \neg(t \to s)$

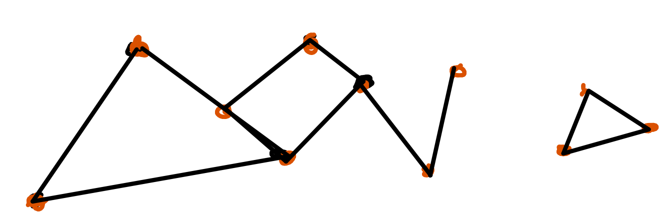
I ideal in $\mathcal{F}_{MV}(k)$ $\quad \mathbb{V}(I) = \bigcap_{t \in I} \mathbb{V}(t)$

$$\mathbb{V}(t) = \{ x \in [0,1]^k \mid t(x) = 0 \}$$

By McNaughton t can be thought of as a continuous piecewise linear map with integer coefficients

the zero-set of a function of the form $\bigwedge_i \bigvee_j a_{ij} x_1 + \dots + a_{ij} x_n + b_{ij}$

The sets of the form $\mathbb{V}(t)$ are called rational polyhedra



$\bullet \in \mathbb{Q}^k$

The sets of the form $\bigcap_{t\in I} V(t)$ are all closed subsets of $[0,1]^k$ with the Euclidean topology.

Theorem Every compact and Hausdorff space can be embedded into some cube $[0,1]^k$ as a closed subspace.

Sketch of the proof: Let X be a topological space consider a family Y_i of top. spaces s.t. $f_i : X \to Y_i$

Define a map

$$ev : X \to \prod Y_i$$
$$x \mapsto (f_i(x))_{i\in I}$$

Kelley's lemma: ev is continuous iff all f_i are continuous
ev is an embedding iff the f_i separate the points
ev is open iff the f_i separate points from closed

Apply this to a compact Hausdorff space X and the family $\mathcal{F}$ of all continuous functions from X into $[0,1]$

$$ev : X \to [0,1]^{\mathcal{F}}$$

Theorem The category of semi-simple MV-algebras is dual to the category of compact Hausdorff spaces with a distinguished embedding into $[0,1]^k$ and cont. piece-wise linear maps with integer coefficients.

Recalling Wojcicki theorem: Finitely presented MV-algebras are semisimple

Corollary The category of finitely presented MV-algebras is dual to the category of rational polyhedra and cont. piece-wise linear maps with integer coefficients.