Lecture 7 9/4/24

Unification concerns with finding a "solution" to an equation $s \approx t$ for s and t terms in some algebraic language $\mathcal{L}$

By a solution we mean a "substitution" σ s.t

$$\sigma(s) = \sigma(t) \qquad \sigma : Free_{\mathcal{L}}(X) \to Free_{\mathcal{L}}(X)$$

The definition can be easily adapted to contemplate equality modulo an equational theory E

$$E \models \sigma(s) = \sigma(t)$$

Usually in unification theory one first study the complexity of the space of solutions (for the worst possible cases)

If σ and σ' are solutions to a unification problem $s \approx t$ we say that σ is more general than σ' if $\exists \tau$

s.t. $\quad E \models \sigma'(x) = \tau \circ \sigma(x) \quad \forall x \in X \qquad [\,\sigma \succcurlyeq \sigma'\,]$

We call mgu (most general unifier) a solution that is maximal in the pre-order of being "more general than".

The possible configurations of mgu's are divided in four cases.

For a unification problem $s \approx t$ there can be.

1) A unique mgu more general than all the other solutions to $s \approx t$

2) A finite number of mgu more general than all the other solutions

3) An infinite " " " " " "

4) Not all solutions are below an mgu.

The unification type of a theory E is one among 1-4 considered in the worst case for E 1) unary, 2) finitary, 3) infinitary

4) nullary.

Ghilardi noticed that the unification type of an equational theory is a categorical property and thus it is preserved under categorical equivalences.

E-Unification problem $\quad\rightsquigarrow\quad$ Finitely presented algebra A in $\mathbb{V}(E)$

$$s \approx t$$

Unifier (solution) $\quad\rightsquigarrow\quad$ Homomorphism from A into a finitely generated projective algebra P.

Remark Finitely generated projective $\Rightarrow$ finitely presented

$$\sigma \succcurlyeq \sigma'$$

$$\begin{tikzcd} P \arrow[d, dashed] & \\ P' & A \arrow[ul, "s"'] \arrow[l, "s'"] \end{tikzcd}$$

$$s \succcurlyeq s'$$

(Marra S.) The unification type of MV-algebras is nullary.

Sketch

MV-Unification problems $\leftrightsquigarrow$ Finitely presented MV-algebras $\leftrightsquigarrow$ Rational polyhedron

MV-solutions $\leftrightsquigarrow$ Morphism into Finitely presented projective MV-algebras $\leftrightsquigarrow$ Morphism coming from Rational polyedra that are retracts of the cubes $[0,1]^n$

$\|$

(Retracts of f.g. free algebras)

Retractions

$$A \xhookrightarrow{s} B \xrightarrow{r} A \qquad r \circ s = id$$

$$A^{op} \xleftarrow{s^{op}} B^{op} \xleftarrow{r^{op}} A^{op} \qquad (r \circ s)^{op} = id^{op}$$

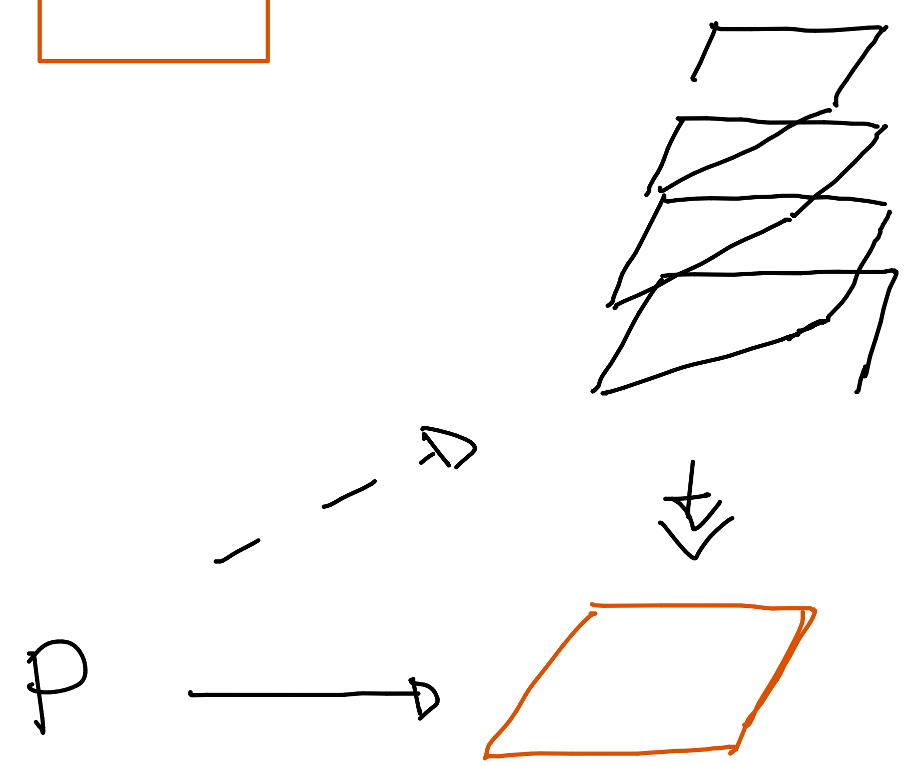
$$\| $$

$$s^{op} \circ r^{op}$$

$$\mathcal{F}_{MV}(k) \longmapsto V(\emptyset) \subseteq [0,1]^k$$

$$V(\emptyset) = [0,1]^k$$

Consider the rational polyhedron given by the border of the unit square $\boxed{B}$. I claim that it has nullary unif. type



$\forall P \ \exists P' \quad P' > P$

Recall that MV-algebras are categorical equivalent to abelian ℓ-groups with a strong order unit.

What happens when we remove the strong unit

Let's go back to ℓ-groups

Let again use the general affine adjunction $\mathbb{V} \dashv \mathbb{I}$

Let set $\quad V =$ Abelian ℓ-groups

$$A = \mathbb{R}$$

Recall that by Hölder theorem any linearly ordered archimedean ℓ-group embeds into $\mathbb{R}$ (and it is simple)

The algebraic fixed points of the adjunction are the semisimple abelian ℓ-groups

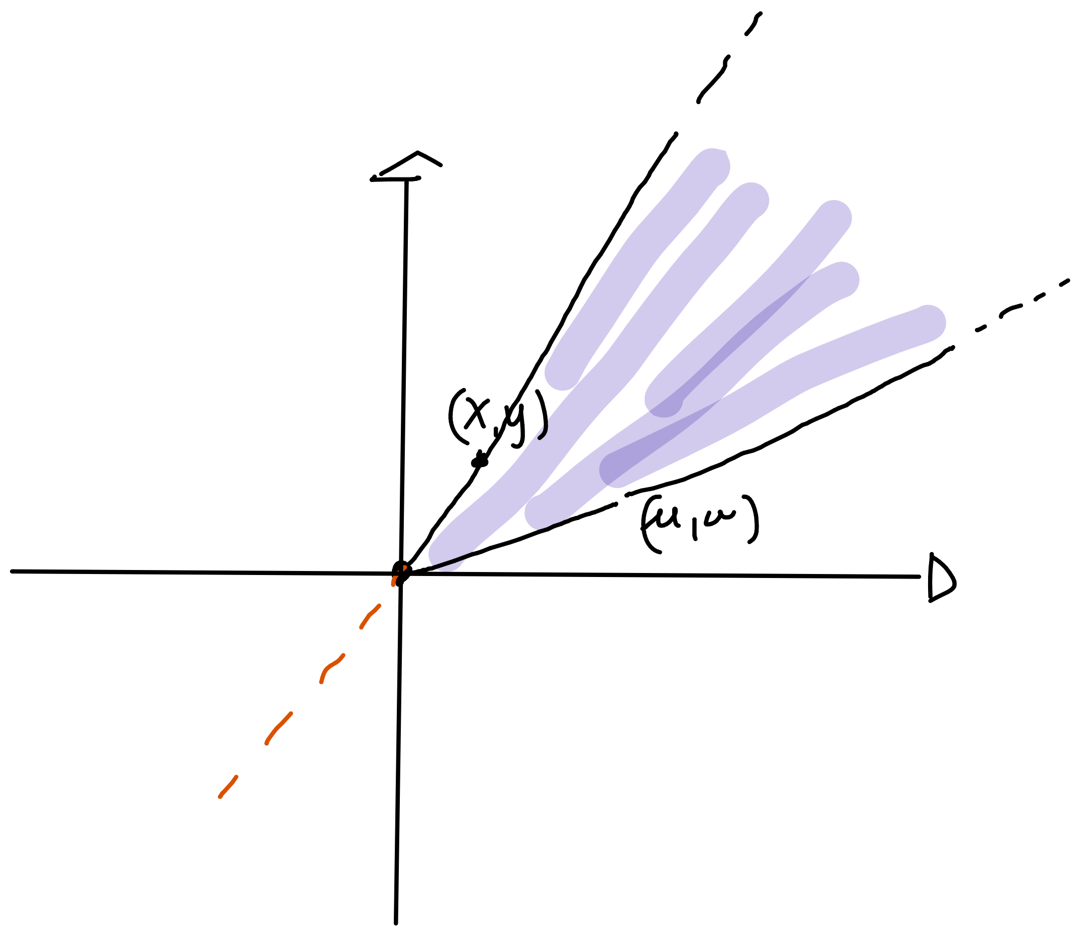
$$\boxed{\begin{array}{l} \theta = \mathbb{I}\,\mathbb{V}(\theta) \iff \theta = \bigcap_{\alpha \in \mathbb{V}(\theta)} \mathbb{I}(\alpha) \\[1em] \theta = \mathbb{I}(\alpha) \qquad \frac{F}{\theta} \hookrightarrow A := \mathbb{R} \end{array}}$$

The geometric fixed points are once more the Zariski closed subsets of $\mathbb{R}^k$. In other words, they are arbitrary intersections of sets of the form

$$\mathbb{V}(f) := \{ x \in \mathbb{R}^k \mid f(x) = 0 \} \quad \text{for } f \in \mathrm{Free}_\ell(k)$$

or equivalently

f piecewise homogeneous map with integer coeff.



$$f(x,y) \geq 0 \qquad \alpha \in \mathbb{R}^+$$

$$ax + by \geq 0 \implies a\alpha x + b\alpha y \geq 0$$

$$f(u,w) \geq 0$$

$$f(x,y) + f(u,w) \geq 0$$

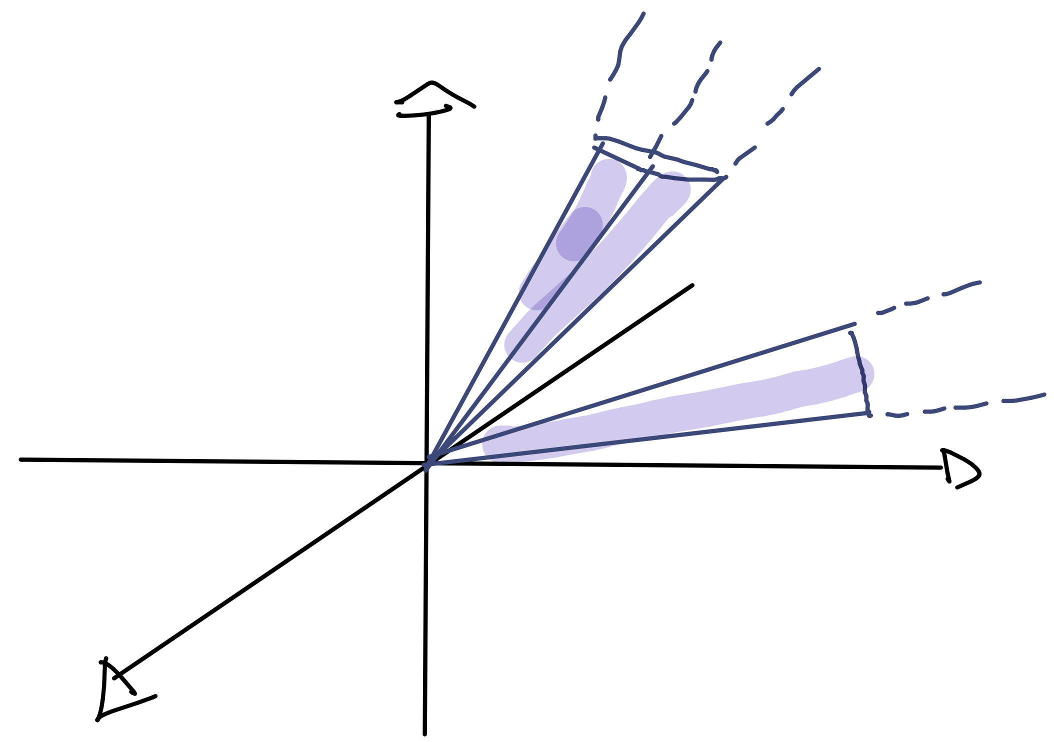
$$f(x+u, y+w) \geq 0$$

Def If $x_1, \dots, x_n \in \mathbb{R}^k$ we call the *positive span* of $x_1 \dots x_n$ the set

$$\{\alpha_1 x_1 + \dots + \alpha_n x_n \mid \alpha_1, \dots, \alpha_n \in \mathbb{R}^+\}$$

Def A *cone* in $\mathbb{R}^k$ is a subset of $\mathbb{R}^k$ that is closed under positive linear combinations. A *closed cone* is a cone that is closed in the euclidean topology. A cone is called *rational* if it is the positive span of vectors with rational coordinates.

In general $\check{V}(f)$ will be a polyhedral rational cone, i.e. a closed cone that is a finite union of the positive spans of finitely many rational vectors.



In general, an arbitrary intersection of such polyhedral rational cones will be a closed cone.

Theorem (Baker & Beynon) The category of semisimple abelian ℓ-groups is dually equivalent to the category of closed cones in $\mathbb{R}^k$ with piecewise homogeneous maps with integer coefficients.