Multivalued Logic

An overview

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Overview

Introduction

Some History

The concept of Fuzzy Logic was conceived by Lotfi Zadeh in 1962, presented not as a control methodology, but as a way of processing data by allowing partial set membership rather than crisp set membership or non-membership.

Zadeh reasoned that people do not require precise, numerical information input, *and yet they are capable of highly adaptive control*.

Some History

A fuzzy subset A of a (crisp) set X is characterized by assigning to each element $x \in X$ the degree of membership of x in A.

Example

X is a group of people and A the fuzzy set of old people in X.

What *Fuzzy* means

- Fuzzy sets, fuzzy logic, fuzzy controls, fuzzy linguistic variable, ...
- Zadeh made precise a notable division in Fuzzy Logic:

  Fuzzy logic in a narrow and in a wide sense

[Zadeh, L.: *Preface, in Fuzzy logic technology and applications*. IEEE Technical Activities Board 1994]

Fuzzy Logic in a wide sense

- Older, better known, heavily applied but not asking deep logical questions
- Serves mainly as apparatus for fuzzy control, analysis of vagueness in natural language and several other application domains.
- It is one of the techniques of soft-computing, i.e. computational methods tolerant to suboptimality and impreciseness (vagueness) and giving quick, simple and sufficiently good solutions

Fuzzy Logic in a narrow sense

- This is symbolic logic with a comparative notion of truth developed fully in the spirit of classical logic:
  - syntax
  - semantics
  - axiomatization
  - truth-preserving deduction
  - etc...
- This fuzzy logic is a relatively young discipline, both serving as a foundation for the fuzzy logic in a broad sense and of independent logical interest

Basic Notions

Hájek's book

A big part of the material presented here can be found in:

[Hájek, P.: *Metamathematics of fuzzy logic*. Kluwer 1998].

Many ideas, results and methods were for the first time introduced in that book.

Starting with the connectives

What kind of assumptions have to be made?

- We want to generalize classical logic, expanding its set of true values.
- The conjunction has to be: commutative, associative, non decreasing.
- If we want a Logic the conjunction needs to be related with the implication

T-norms

T-norms...

Definition

A t-norm $*$ is a function from $[0,1]^2$ to $[0,1]$ that is

- associative and commutative
- non-decreasing in both argument, i.e $x_1 \leq x_2$ implies $x_1 * y \leq x_2 * y$ and $x_1 \leq x_2$ implies $y * x_1 \leq y * x_2$
- $1 * x = x$ and $x * 0 = 0$

...and their residua

Definition

Let $*$ be a continuous t-norm. The unique opertation $x \Rightarrow y$ satisfying the following condition:

$$(x * z \leq y) \text{ if and only if } z \leq (x \Rightarrow y)$$

namely: $x \Rightarrow y = max\{z \mid x * z \leq y\}$ is called the residuum of $*$

Examples

- Łukasiewicz t-norm: $x * y = max\{0, x + y - 1\}$; and its residuum $x \Rightarrow y = min\{1, 1 - x + y\}$
- Gödel t-norm: $x * y = min\{x, y\}$; and its residuum
  $$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$
- Product t-norm: $x * y = x \cdot y$; and its residuum
  $$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$$

Remark. The above three norms form a complete system in the sense that every other t-norm is locally isomorphic to one of them

Back to the logic

If we fix a t-norm we fix a Logic calculus interpreting the t-norm as the truth function of the conjunction

Definition

The propositional calculus PC(*) has propositional variables $p_1, \ldots, p_n, \ldots$, connectives & and $\rightarrow$. Formulas are built as usual. Further connectives are defined:

- $\varphi \wedge \psi = \varphi \& (\varphi \rightarrow \psi)$
- $\varphi \vee \psi = (\varphi \rightarrow \psi) \rightarrow \psi \wedge (\psi \rightarrow \varphi) \rightarrow \varphi$
- $\neg \varphi = \varphi \rightarrow 0$
- $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$

Evaluation

Definition

An evaluation e is a function from propositional variables to $[0,1]$. It extends in a unique way to the formulas according to the following constraints:

- $e(0) = 0$
- $e(\varphi \& \psi) = e(\varphi) * e(\psi)$
- $e(\varphi \to \psi) = e(\varphi) \Rightarrow e(\psi)$

Definition

A formula φ of PC($*$) is a 1-tautology iff for any evaluation one has $e(\varphi) = 1$

BL logic

Definition

The following are the axioms of BL Logic

- $(\varphi \to \psi) \to ((\psi \to \theta) \to (\varphi \to \theta))$
- $(\varphi \& \varphi) \to \varphi$
- $(\varphi \& \psi) \to (\psi \& \varphi)$
- $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$
- $(\varphi \to (\psi \to \theta)) \to ((\varphi \& \psi) \to \theta)$
- $((\varphi \to \psi) \to \theta) \to ((\psi \to \varphi) \to \theta) \to \theta)$
- $0 \to \varphi$

Deduction theorem

Theorem

Let T be a theory and φ, ψ formulas. $T \cup \{\varphi\} \vdash \psi$ iff there is an n s.t. $T \vdash \varphi^n \rightarrow \psi$

- To better understand the Logic we try to give it an algebraic semantic
- This is made in the same spirit of Classical Logic

BL algebras

Definition

A residuated lattice is a structure $\mathcal{A} = \langle A, *, \Rightarrow, \wedge, \vee, 0, 1 \rangle$

- $\langle A, \wedge, \vee, 0, 1 \rangle$ is a lattice with greatest and least element being repsectively 1 and 0
- $\langle A, *, 1 \rangle$ is a commutative monoid
- $*$ and $\Rightarrow$ form an adjoint pair, i.e. $z \leq (x \Rightarrow y)$ iff $x * z \leq y$

Definition

A residuated lattice $\mathcal{A} = \langle A, *, \Rightarrow, \wedge, \vee, 0, 1 \rangle$ is a BL algebra if it satisfies

- $x \wedge y = x * (x \wedge y)$ (divisibility)
- $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ (pre-linearity)

Lindenbaum-Tarski algebra

Definition

Let T be a theory over BL. For each formula φ let $[\varphi]_T$ be the set of formula ψ such that $T \vdash \psi \leftrightarrow \varphi$. Then define

- $0 = [0]_T$
- $1 = [1]_T$
- $[\varphi]_T * [\psi]_T = [\varphi \& \psi]_T$
- $[\varphi]_T \Rightarrow [\psi]_T = [\varphi \to \psi]_T$
- $[\varphi]_T \cap [\psi]_T = [\varphi \wedge \psi]_T$
- $[\varphi]_T \cup [\psi]_T = [\varphi \vee \psi]_T$

This algebra will be denoted as $\mathbf{L}_T$

Some Lemma

Lemma

$\mathbf{L}_T$ *is a BL algebra*

Definition

Given a lattice L, a filter F is a non empty subset of L s.t.

$$\text{If } a, b \in F \text{ then } a \cap b \in F$$

$$\text{If } a \in F \text{ and } a \leq b \text{ then } b \in F$$

A filter is said to be prime if for any $x, y \in L$ either $(x \Rightarrow y) \in F$ or $(y \Rightarrow x) \in F$

Some Lemma

Lemma

Let L be a BL algebra and F a filter. Put

$$x \sim_F y \text{ iff } (x \Rightarrow y) \in F \text{ and } (y \Rightarrow x) \in F$$

then

- $\sim_F$ *is a congruence and the corresponding quotient* $L/\sim_F$ *is a BL algebra*
- $L/\sim_F$ *is linearly ordered iff* F *is prime*

Some Lemma

Lemma

Let L be a BL algebra and $a \in L$, with $a \neq 1$, then there is a prime filter not containing a

Lemma

Every BL algebra is the subdirect product of linearly ordered BL algebra

Completeness

Theorem

BL algebras are the algebraic semantic for BL logic. In other words whenever a formula ϕ is provable in the logic BL, then it holds in every BL algebra

Theorem (Cignoli et al.)

BL is the logic of all continuous t-norm. In other words whenever a formula ϕ is provable in the logic BL, then it holds for every t-norm $*$

Importance of BL

BL logic is hence important in two senses

- It gives us a formal system to prove properties that are common to all t-norms.
- It is an important logic because it generalizes the above mentioned three most important t-norm based logics. Indeed one can rescue any of the three logic calculi just adding one (or two) axioms to BL system

Łukasiewicz Logic

The system Ł

Definition (old style)

Łukasiewicz Logic has the following axioms:

- $\varphi \to (\psi \to \varphi)$
- $(\varphi \to \theta) \to (\theta \to \psi) \to (\varphi \to \psi)$
- $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
- $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

Definition

Łukasiewicz Logic is BL Logic plus the following axiom:

$$x = (x \to 0) \to 0$$

MV algebra

Definition

A Łukasiewicz algebra (bka. MV algebra or Wejsbergh algebra, or ...) is a BL algebra that satisfies the following axiom

$$x = (x \Rightarrow 0) \Rightarrow 0$$

Results about Ł

Theorem (Completeness)

The Logic Ł is sound and complete w.r.t the class of MV algebras.

Definition

Let $\mathcal{G} = \langle G, +, \leq, 0\rangle$ be a linearly ordered abelian group (o-group for short) and let e be a positive element. $MV(G, e)$ is the algebra $\mathcal{A} = \langle A, \Rightarrow, 0\rangle$ whose domain is $A = [0, e]$ and

$$x \Rightarrow y = \begin{cases} e & \text{if } x \leq y \\ e - x + y & \text{otherwise} \end{cases}$$

Results about Ł. Cont'd

Theorem (Representation)

There is a categorical equivalence between lo-groups with strong unit and MV algebra.

Theorem (Strong Completeness)

The Łis standard complete. In other words, whenever a formula φ is true in $[0,1]_L$ it can be proved in Ł.

Göedel Logic

The system G

Definition

A Göedel algebra is a BL algebra that satisfies the following axiom

$$x = x \wedge x$$

Results about G

Theorem (Completeness)

The Logic G is sound and complete w.r.t the class of Heiting algebras satisfying prelinearity.

Theorem (Strong Completeness)

The G is standard complete. In other words, whenever a formula φ is true in $[0,1]_G$ it can be proved in G.

Product Logic

The system Π

Definition

A Π algebra is a BL algebra that satisfies the following axiom

- $\neg\neg\theta \rightarrow ((\varphi * \theta \rightarrow \psi * \theta) \rightarrow (\varphi \rightarrow \psi))$
- $\varphi \wedge \neg\varphi \rightarrow 0$

Results about Π

Theorem (Completeness)

The Logic Π is sound and complete w.r.t the class of Π algebras.

Theorem (Strong Completeness)

The Π is standard complete. In other words, whenever a formula φ is true in $[0,1]_\Pi$ it can be proved in Π.

Results about Π. Cont'd

Theorem (Representation)

Theorem (Strong Completeness)

The Π is standard complete. In other words, whenever a formula φ is true in $[0,1]_\Pi$ it can be proved in Π.

Advanced Topic

Compactness

Generalizing the Concept

It is natural to add more degree of satisfiability to the formulas

Definition

Let Γ be a set of formula in a Fuzzy Logic and $K \subseteq [0,1]$. We say that Γ is K-satisfiable if there exist an evaluation e s.t. $e(\varphi) \in K$ for any $\varphi \in K$. We say that a Logic is K-compact if finitely K-satisfiable is equivalent to K-satisfiable. We say that a logic satisfies the compactness property if it is K-compact for any K closed set.

Definition

A nonempty subset K of $[0,1]$ is of type C if $0 \notin K$ or $1 \notin K$. Furthermore, if K is of type C we define other types: C_0 if $0 \in K$, C_1 if 1 K , $C_{\bar{0}}$ if $0 \notin K$, $C_{\bar{1}}$ if $1 \notin K$, C_{01} if $0 \in K$ or $1 \in K$

Positive Results

- Łukasiewicz Logic has the compactness property
- Łukasiewicz Logic is not K-compact for any non compact subset K of $[0, 1]$ of type C
- If K is of type C_1 then Göedel and Product Logic are K-compact

Negative Results

- Göedel Logic does not satisfies the compactness property
- Product Logic does not satisfies the compactness property
- ... and many others

ŁΠ logic

The system ŁΠ

Definition

The Logic ŁΠ is axiomatized as following

- All the axioms of Ł
- All the axioms of Π
- k

Proof Theory

Proof Theory

No time for it!

Consult George Metcalfe's page

http://www.dcs.kcl.ac.uk/pg/metcalfe/