Fuzzy Logic and Algebra

An overview

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Overview

The mathematical core of Fuzzy Logic

- Fuzzy Logic has undoubtedly gained an important role in engineering and industry.

  This is due to its **flexibility** and **feasibility**.

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- Fuzzy Logic has undoubtedly gained an important role in engineering and industry.

  This is due to its **flexibility** and **feasibility**.
- But it lacks a solid **mathematical background**.

  The aim is to give stromg mathematical/logical foundations

  To this end we start back from the core of the logic.

Starting with the connectives

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- The conjunction has to be: **commutative**, **associative** and **non decreasing** in both arguments.
- If we want a Logic the **conjunction** needs to be **related** with the **implication**

T-norms and their residua

Definition

A t-norm $*$ is a function from $[0,1]^2$ to $[0,1]$ that is

- $1 * x = x$ and $x * 0 = 0$
- associative and commutative
- non-decreasing in both argument, i.e $x_1 \leq x_2$ implies $x_1 * y \leq x_2 * y$ and $x_1 \leq x_2$ implies $y * x_1 \leq y * x_2$

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Definition

Let $*$ be a continuous t-norm. The unique opertation $x \Rightarrow y$ satisfying the following condition:

$$(x * z) \leq y \text{ if and only if } z \leq (x \Rightarrow y)$$

namely: $x \Rightarrow y = max\{z \mid x * z \leq y\}$ is called the residuum of $*$

Examples

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  $$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

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Remark. The above three functions form a **complete system** in the sense that every other t-norm is locally isomorphic to them.

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- $\neg\varphi = \varphi \rightarrow 0$
- $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi)\&(\psi \rightarrow \varphi)$
- $\varphi \wedge \psi = \varphi\&(\varphi \rightarrow \psi)$
- $\varphi \vee \psi = (\varphi \rightarrow \psi) \rightarrow \psi \wedge (\psi \rightarrow \varphi) \rightarrow \varphi$

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Definition

A formula φ of PC($*$) is a 1-tautology iff for any evaluation one has $e(\varphi) = 1$

BL logic

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Definition

The following are the axioms of BL Logic

- $(\varphi \to \psi) \to ((\psi \to \theta) \to (\varphi \to \theta))$
- $(\varphi \& \varphi) \to \varphi$
- $(\varphi \& \psi) \to (\psi \& \varphi)$
- $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$
- $(\varphi \to (\psi \to \theta)) \to ((\varphi \& \psi) \to \theta)$
- $((\varphi \to \psi) \to \theta) \to ((\psi \to \varphi) \to \theta) \to \theta)$
- $0 \to \varphi$

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- $\langle A, \wedge, \vee, 0, 1 \rangle$ is a lattice with greatest and least element being repsectively 1 and 0
- $\langle A, *, 1 \rangle$ is a commutative monoid
- $*$ and $\Rightarrow$ form an adjoint pair, i.e. $z \leq (x \Rightarrow y)$ iff $x * z \leq y$

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Definition

A residuated lattice $\mathcal{A} = \langle A, *, \Rightarrow, \wedge, \vee, 0, 1 \rangle$ is a BL algebra if it satisfies

- $x \wedge y = x * (x \Rightarrow y)$ (divisibility)
- $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ (pre-linearity)

Lindenbaum-Tarski algebra

Definition

Let T be a theory over BL. For each formula φ let $[\varphi]_T$ be the set of formula ψ such that $T \vdash \psi \leftrightarrow \varphi$.

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- $0 = [0]_T$
- $1 = [1]_T$
- $[\varphi]_T * [\psi]_T = [\varphi \& \psi]_T$
- $[\varphi]_T \Rightarrow [\psi]_T = [\varphi \rightarrow \psi]_T$
- $[\varphi]_T \cap [\psi]_T = [\varphi \wedge \psi]_T$
- $[\varphi]_T \cup [\psi]_T = [\varphi \vee \psi]_T$

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Lemma

$\mathbf{L}_T$ *is a BL algebra*

Some Lemma

Definition

Given a lattice L, a filter F is a non empty subset of L s.t.

$$\text{If } a, b \in F \text{ then } a \cap b \in F$$

$$\text{If } a \in F \text{ and } a \leq b \text{ then } b \in F$$

A filter is said to be prime if for any $x, y \in L$ either $(x \Rightarrow y) \in F$ or $(y \Rightarrow x) \in F$

Some Lemma

Lemma

Let L be a BL algebra and F a filter. Let $x \sim_F y$ if, and only if, $(x \Rightarrow y) \in F$ and $(y \Rightarrow x) \in F$ then

- *$\sim_F$ is a congruence and the corresponding quotient $L/\sim_F$ is a BL algebra*
- *$L/\sim_F$ is linearly ordered iff F is prime*

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Lemma

Let L be a BL algebra and $a \in L$, with $a \neq 1$, then there is a prime filter not containing a

Completeness

Theorem

Every BL algebra is the subdirect product of linearly ordered BL algebras

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Theorem (Cignoli et al.)

BL is the logic of all continuous t-norm. In other words a formula φ is provable in the logic BL if, and only if, it holds for every t-norm $*$

Summing up

BL logic is hence important for two reasons

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1. It gives us a formal system to prove properties that are common to all t-norms.
2. It generalizes the above mentioned three most important t-norm based logics. Indeed one can rescue any of the three logical systems just by adding one axiom to BL.

The system G

Reminder

Gödel t-norm is defined as:

$$x * y = min\{x, y\}$$

and its residuum

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

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$$x = x * x$$

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Theorem (Completeness)

The Logic G is sound and complete w.r.t the class of Heiting algebras satisfying prelinearity.

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Theorem (Standard Completeness)

The G is standard complete. In other words, a formula φ is true in $[0,1]_G$ if, and only if, it can be proved in G.

The system Π

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Product t-norm is defined as:

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Definition

A Π algebra is a BL algebra that satisfies the following axiom

$$(y \Rightarrow 0) \vee ((y \Rightarrow x * y) \Rightarrow x)$$

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The Π is standard complete. In other words, whenever a formula φ is true in $[0,1]_\Pi$ it can be proved in Π.

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Reminder

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Definition (old style)

Łukasiewicz Logic has the following axioms:

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- $(\varphi \to \theta) \to (\theta \to \psi) \to (\varphi \to \psi)$
- $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
- $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

MV algebra

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Theorem (Completeness)

The Logic Ł is sound and complete w.r.t the class of MV algebras.

Theorem (Standard Completeness)

The calculus Ł is standard complete. In other words, a formula φ is true in $[0,1]_L$ if, and only if, it can be proved in Ł.

Results about Ł

Definition

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- $\langle G, +, -, 0 \rangle$ is an abelian group
- $\langle G, \vee, \wedge \rangle$ is a lattice
- If $\leq$ denotes the partial order given by $\wedge, \vee$ then: if $x \leq y$ then $x + z \leq y + z$

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- If $\leq$ denotes the partial order given by $\wedge, \vee$ then: if $x \leq y$ then $x + z \leq y + z$
- For any $x \in G$ there is $n \in \mathbb{N}$ such that $\underbrace{1 + ... + 1}_{\text{n times}} \geq x$

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Theorem (Representation)

There is a categorical equivalence between lattice ordered groups with strong unit and MV algebras.

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2. There are important links with other fields of mathematics
3. These links are important to prove standard completeness but they are also interesting in their own.

Fixed points

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(To start with) we chose the **most expressive** among t-norm based logic

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- $\varphi \&_{\Pi} (\psi \ominus \theta) \leftrightarrow_{L} (\varphi \&_{\Pi} \psi) \ominus (\varphi \&_{\Pi} \theta)$

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- The rules *Modus Ponens* and *Necessitation* $\frac{\varphi}{\Delta(\varphi)}$

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Theorem

ŁΠ logic faithful interprets Ł, Π and G.

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3. If $\varphi(p) \leftrightarrow p$ then $\mu x.\varphi(x) \rightarrow p$

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2. $\mu x.\varphi(x) \leftrightarrow \varphi(\mu x.\varphi(x))$
3. If $\varphi(p) \leftrightarrow p$ then $\mu x.\varphi(x) \rightarrow p$
4. If $\bigwedge_{i\leq n} (p_i \leftrightarrow q_i)$ then $\mu x.\varphi(p_1, ..., p_n) \leftrightarrow \mu x.\varphi(q_1, ..., q_n)$

Results on ŁΠ with fixed points

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Theorem

The category of μŁΠ algebras and the category of subdirect products of real closed fields are equivalent.

Suggested reading

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