Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras

Vincenzo Marra[a], Luca Spada[b]

[a]Dipartimento di Informatica e Comunicazione, Università degli Studi di Milano. Via Comelico 39–41, 20135 Milano, Italy
[b]Dipartimento di Matematica, Università degli Studi di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy

Abstract

We prove that the unification type of Łukasiewicz (infinite-valued propositional) logic and of its equivalent algebraic semantics, the variety of MV-algebras, is nullary. The proof rests upon Ghilardi's algebraic characterisation of unification types in terms of projective objects, recent progress by Cabrer and Mundici in the investigation of projective MV-algebras, the categorical duality between finitely presented MV-algebras and rational polyhedra, and, finally, a homotopy-theoretic argument that exploits lifts of continuous maps to the universal covering space of the circle. We discuss the background to such diverse tools. In particular, we offer a detailed proof of the duality theorem for finitely presented MV-algebras and rational polyhedra — a fundamental result that, albeit known to specialists, seems to appear in print here for the first time.

Key words: Łukasiewicz logic, Unification, Projective MV-algebras, Rational Polyhedra, Retractions, Fundamental group, Covering space, Universal cover, Lifts.
2010 MSC: Primary: 06D35. Secondary: 03B50, 03C05, 57M10, 52B20.

1. Introduction.

The origins of the theory of unification are usually traced back to the doctoral thesis that Herbrand defended at the Sorbonne in the summer of 1930; an annotated English translation of its final Chapter 5 is available in [32, pp. 525–581]. It was only with Robinson's landmark paper [27] on resolution, however, that the first unification algorithm with termination and correctness proofs appeared in print. Unification has attracted continuing interest to this day as a basic tool in automated deduction. The study of unification modulo an equational theory that grew out of such pioneering works as [26] has acquired increasing significance in recent years; see the extensive survey [3], and also [2] for a more recent survey focused on modal logic. The classical, syntactic unification problem is: given two terms s, t (built from function symbols and variables), find a *unifier* for them, that is, a uniform replacement of the variables occurring in s and t by other terms that makes s and t identical. When the latter syntactical identity is replaced by equality modulo a given equational theory E, one speaks of *E-unification*. Unsurprisingly, E-unification can be far harder than syntactic unification even when the theory E comes from the least exotic corners of the mathematical world. For instance, it may well be impossible to uniformly decide whether two terms admit at least one unifier, i.e. whether they are *unifiable* at all; and even when the two terms indeed are unifiable, there may well be no most general unifier for them, contrary to the situation in the syntactic case. In light of these considerations, perhaps the most basic piece of information one would like to have about E in connection with unification issues is its *unification type*.[1] In order to define it precisely, let us recall some standard notions.

We consider a set $\mathscr{F}$ of function symbols — the *signature* — along with a further set $\mathscr{V}$ — the *variables*; each function symbol comes with its own arity, an integer $n \geqslant 0$, with $n = 0$ being included to accommodate

Email addresses: marra@dico.unimi.it (Vincenzo Marra), lspada@unisa.it (Luca Spada)

[1]Strictly speaking, throughout this paper we are concerned with the *elementary unification type of E*, meaning that in unification problems and unifiers we do not allow terms with additional function symbols not included in the signature $\mathscr{F}$ of E; see [3, Definition 3.9].

Preprint submitted to APAL on Tuesday 19th April, 2011

constants. It is usual to require that $\mathscr{V}$ be countable. Let us therefore set, once and for all throughout the paper,

$$\mathscr{V} = \{X_1, X_2, \ldots\}.$$

We then let $\mathsf{Term}_{\mathscr{V}}(\mathscr{F})$ be the *term algebra* built from $\mathscr{F}$ and $\mathscr{V}$ in the usual manner [5, Definition 10.1]. A *substitution*[2] is a mapping $\sigma\colon \mathscr{V} \to \mathsf{Term}_{\mathscr{V}}(\mathscr{F})$ that acts identically to within a finite number of exceptions, i.e. is such that $\{X \in \mathscr{V} \mid \sigma(X) \neq X\}$ is a finite set. Substitutions compose in the obvious manner. By an *equational theory* over the signature $\mathscr{F}$ one means a set $E = \{(l_i, r_i) \mid i \in I\}$ of pairs of terms $l_i, r_i \in \mathsf{Term}_{\mathscr{V}}(\mathscr{F})$, where I is some index set. The set of equations E axiomatises the *variety of algebras* [5, Theorem 11.9] consisting of the models of the theory E, written $\mathbb{V}_E$.

Now a (*symbolic*) *unification problem modulo* E is a finite set of pairs

$$\mathscr{E} = \{(s_j, t_j) \mid s_j, t_j \in \mathsf{Term}_{\mathscr{V}}(\mathscr{F}) \ , \ j \in J\},$$

for some finite index set J. A *unifier* for $\mathscr{E}$ is a substitution σ such that

$$E \models \sigma(s_j) \approx \sigma(t_j),$$

for each $j \in J$, i.e. such that the equality $\sigma(s_j) = \sigma(t_j)$ holds in every algebra of the variety $\mathbb{V}_E$ in the usual universal-algebraic sense [5, p. 78]. The problem $\mathscr{E}$ is *unifiable* if it admits at least one unifier. The set $U(\mathscr{E})$ of unifiers for $\mathscr{E}$ can be partially ordered as follows. If σ and τ are substitutions and $V \subseteq \mathscr{V}$ is a set of variables, we say that σ is *more general*[3] than τ (with respect to E and V), written $\tau \preceq^V_E \sigma$, if there exists a substitution ρ such that

$$E \models \tau(X) \approx (\rho \circ \sigma)(X)$$

holds for every $X \in V$. This amounts to saying that τ is an instantiation of σ, but only to within E-equivalence, and only as far as the set of variables V is concerned. We endow $U(\mathscr{E})$ with the relation $\preceq^V_E$, where V is the set of variables occurring in the terms s_j, t_j with $(s_j, t_j) \in \mathscr{E}$, as j ranges in J. The relation $\preceq^V_E$ is a pre-order. There is an equivalence relation $\sim$ on $U(\mathscr{E})$ that identifies σ and τ if and only if $\tau \preceq^V_E \sigma$ and $\sigma \preceq^V_E \tau$ both hold. The quotient set $\frac{U(\mathscr{E})}{\sim}$ carries the canonical partial order $\leqslant^V_E$ associated to the pre-order $\preceq^V_E$; by definition, $[\sigma] \leqslant^V_E [\tau]$ if and only if $\sigma \preceq^V_E \tau$, where $[\sigma]$ and $[\tau]$ respectively denote the equivalence classes induced by $\sim$ of the unifiers σ and τ. We call $\left(\frac{U(\mathscr{E})}{\sim}, \leqslant^V_E\right)$ the *partially ordered set of unifers for* $\mathscr{E}$, even though its elements actually are equivalence classes of unifiers.

The (*symbolic*) *unification type* of the unification problem $\mathscr{E}$ is:

- ▷ *unitary*, if $\leqslant^V_E$ admits a maximum $[\mu] \in \frac{U(\mathscr{E})}{\sim}$;
- ▷ *finitary*, if $\leqslant^V_E$ admits no maximum, but admits finitely many maximal elements $[\mu_1], \ldots, [\mu_u] \in \frac{U(\mathscr{E})}{\sim}$ such that every $[\sigma] \in \frac{U(\mathscr{E})}{\sim}$ lies below some $[\mu_i]$;
- ▷ *infinitary*, if $\leqslant^V_E$ admits infinitely many maximal elements $\left\{[\mu_i] \in \frac{U(\mathscr{E})}{\sim} \mid i \in I\right\}$, for I an infinite index set, such that every $[\sigma] \in \frac{U(\mathscr{E})}{\sim}$ lies below some $[\mu_i]$;
- ▷ *nullary*, if none of the preceding cases applies.

---

[2]It would be more expedient to define unifiers for $\mathscr{E}$ as substitutions having a finite domain coincident with the set of variables occurring in $\mathscr{E}$. This would perfectly match the definition of the pre-order $\preceq^V_E$ on unifiers recalled below, which only compares unifiers on those variables occurring in $\mathscr{E}$. We have nonetheless chosen to follow [3] in the basic definitions in order not to depart from established practice; cf. [3, 3.2.1] for a related discussion.

[3]The convention adopted in [3] is that '$\tau \preceq^V_E \sigma$' means 'τ is more general than σ', whereas here we are following [14] in choosing the opposite reading.

It is understood that the list above is arranged in decreasing order of desirability. In the best, unitary case, any element of the maximum equivalence class $[\mu]$ is called a *most general unifier* for $\mathscr{E}$, or *mgu* for short. An mgu is then unique up to the relation $\sim$, whence one speaks of *the* mgu for $\mathscr{E}$. If $[\mu]$, on the other hand, is maximal but not a maximum, then any element of $[\mu]$ is called a *maximally general unifier*.

The *unification type* of the equational theory E is now defined to be the worst unification type occurring among the unification problems $\mathscr{E}$ modulo E.

This paper is devoted to an investigation of the unification type of *Łukasiewicz* (*infinite-valued propositional*) *logic*, a non-classical system going back to the 1920's (cf. the early survey [20, §3], and its annotated English translation in [31, pp. 38–59]). Łukasiewicz logic may be axiomatised using the primitive connectives $\to$ (implication) and $\neg$ (negation) by the four axiom schemata:

(A1) $\alpha \to (\beta \to \alpha)\,,$

(A2) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))\,,$

(A3) $(\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)\,,$

(A4) $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)\,,$

with *modus ponens* as the only deduction rule. The semantics of Łukasiewicz logic is many-valued: assignments μ to atomic formulæ range in the unit interval $[0, 1] \subseteq \mathbb{R}$; they are extended compositionally to compound formulæ via

$$\mu(\alpha \to \beta) = \min\{1, 1 - \mu(\alpha) + \mu(\beta)\}\,,$$
$$\mu(\neg\alpha) = 1 - \mu(\alpha)\,.$$

Tautologies are defined as those formulæ that evaluate to 1 under every such assignment, and contradictions are therefore formulæ that constantly evaluate to 0. Completeness of the axioms (A1–A4) with respect to this semantics was established by syntactic means in [28]. Chang [8] first considered the Tarski-Lindenbaum algebras of Łukasiewicz logic, and called them *MV-algebras*. In [9] he obtained an algebraic proof of the completeness theorem.

Thanks to almost a century of hindsight, it is by now apparent that Łukasiewicz's terse formal system relates strongly to several traditional fields of mathematics. The standard reference for the elementary theory is [10], whereas [25] deals with topics at the frontier of current research. Let us recall that an MV-algebra is an algebraic structure $(M, \oplus, \neg, 0)$, where $0 \in M$ is a constant, $\neg$ is a unary operation satisfying $\neg\neg x = x$, $\oplus$ is a unary operation making $(M, \oplus, 0)$ a commutative monoid, the element 1 defined as $\neg 0$ satisfies $x \oplus 1 = 1$, and the law

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \tag{*}$$

holds. Any MV-algebra has an underlying structure of distributive lattice bounded below by 0 and above by 1. Joins are defined as $x \vee y = \neg(\neg x \oplus y) \oplus y$. Thus, the characteristic law (*) states that $x \vee y = y \vee x$. Meets are defined by the de Morgan condition $x \wedge y = \neg(\neg x \vee \neg y)$. To recover the algebraic counterpart of Łukasiewicz implication from the MV-algebraic signature, set $x \to y = \neg x \oplus y$. Conversely, the logical counterpart of the monoidal operation $\oplus$ is definable in Łukasiewicz logic as $\alpha \oplus \beta = \neg\alpha \to \beta$. The algebraic constants 0 and $1 = \neg 0$ respectively correspond to an arbitrary but fixed contradiction and tautology of the logic. Boolean algebras are precisely those MV-algebras that are idempotent, meaning that $x \oplus x = x$ holds, or equivalently, that satisfy the *tertium non datur* law $x \vee \neg x = 1$.

Our main result is the following

Theorem. *The unification type of the variety of MV-algebras is nullary. Specifically, consider the unification problem in the language of MV-algebras*

$$\mathscr{E} = \{\, (X_1 \vee \neg X_1 \vee X_2 \vee \neg X_2\,,\, 1)\,\}\ . \tag{$\star$}$$

Then the partially ordered set of unifiers for $\mathscr{E}$ contains a co-final chain of order-type ω.

We recall that a subset C of a partially ordered set $(P, \leqslant)$ is *co-final* if for every $p \in P$ there is $c \in C$ with $p \leqslant c$. By a *chain of order-type* ω we mean, as usual, a totally ordered set that is order-isomorphic to the natural numbers with their natural order. In particular, the Theorem implies that *no unifier for the unifiable problem* $\mathscr{E}$ *is maximally general* — a condition that is strictly stronger than nullarity.

Because MV-algebras are the equivalent algebraic semantics for Łukasiewicz logic, in the precise sense of Blok and Pigozzi [13], the Theorem easily entails an analogous statement for Łukasiewicz logic. In the sequel we shall concentrate on the MV-algebraic formulation above.

The interval (of truth values) $[0,1] \subseteq \mathbb{R}$ can be made into an MV-algebra with neutral element 0 by defining $x \oplus y = \min\{x+y, 1\}$ and $\neg x = 1 - x$. The underlying lattice order of this MV-algebra coincides with the natural order that $[0,1]$ inherits from the real numbers. Each assignment $\mu\colon \{X_1, X_2\} \to [0,1]$ can be identified with the point $(\mu(X_1), \mu(X_1)) \in [0,1]^2$ lying in the square $[0,1]^2$. Moreover, the set of those assignments μ such that $\mu(X_1 \vee \neg X_1 \vee X_2 \vee \neg X_2) = \mu(1)$ is precisely the boundary $\mathfrak{B}$ of $[0,1]^2$; indeed, for any two terms s and t in the language of MV-algebras, on the preceding definitions we have $\mu(s \vee t) = \max\{\mu(s), \mu(t)\}$, $\mu(\neg s) = 1 - \mu(s)$, and $\mu(1) = 1$. If $[0,1]^2$ is endowed with its Euclidean metric topology, then $\mathfrak{B}$ inherits a subspace topology that makes it homeomorphic to $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, the standard unit circle in the plane. In particular, $\mathfrak{B}$ is not simply-connected: it is connected, but its fundamental (Poincaré) group is not trivial. It will transpire from the proof of the Theorem that this property of $\mathfrak{B}$ is the deeper reason why $(\star)$ has nullary type. To bring algebraic topology to bear on the proof, however, several diverse tools must be used; we will provide some background as needed. Let us now discuss the outline of the paper.

Section 2 contains preliminaries. After a brief reminder on MV-algebras (Subsection 2.1) and on projective objects (Subsection 2.2), in Subsection 2.3 we summarise Ghilardi's approach to E-unification through projectivity [14]. A unification problem $\mathscr{E}$ as in the above is modelled by the algebra finitely presented by the relations $s_j = t_j$, and a unifier is modelled by a homomorphism $u\colon A \to P$, with P a finitely presented projective algebra P. Unifiers are pre-ordered via comparison arrows. Ghilardi's main general result [14, Theorem 4.1], quoted here as Theorem 2.1, is that the algebraic unification type defined along these lines coincides with the traditional, symbolic unification type, at least for varieties with finite signature. This important fact underlies our whole paper.

Coupling algebraic unification with Stone-type dualities often leads to decisive topological insight. In [14, Theorem 5.7], for instance, Ghilardi used the basic duality between finitely presented (=finite) distributive lattices and finite partially ordered sets to show that the unification type of distributive lattices is nullary. Here, too, we will dualise the unification problem $(\star)$ to prove the Theorem. Specialists know that the full subcategory of finitely presented MV-algebras is dually equivalent to a category of rational polyhedra whose morphisms, called $\mathbb{Z}$-*maps*, are continuous functions with additional properties. Since no published version of this duality theorem seems to be available, we offer an essentially self-contained proof in Theorem 3.4 of Section 3. Subsection 2.4 contains the required background on polyhedral geometry. In the rest of this paper we refer to Theorem 3.4 simply as 'the duality theorem'.

With the duality theorem at our disposal, it is an easy matter to obtain in Subsection 3.4 a dual description of projective finitely presented MV-algebras. The dual rational polyhedra are precisely those obtainable as retracts of unit cubes $[0,1]^n$ by $\mathbb{Z}$-maps, for some positive integer n; it follows that they are simply-connected — unlike $\mathfrak{B}$, the one associated with the unification problem $(\star)$. The intrinsic characterisation of such retracts (equivalently, of finitely presented projective MV-algebras) is a non-trivial open problem. Nonetheless, sufficient information for our purposes is already available thanks to the important advances achieved by Cabrer and Mundici in [7]. The needed result is quoted in Subsection 3.4 as Lemma 3.8.

In Section 4 we show (Theorem 4.1) that 1-variable unification problems in the language of MV-algebras always have at most two maximally general unifiers that dominate all other unifiers. Although this special case is relatively simple, it is included here by way of warm-up: the proof uses the same tools applied in the general case, with the single exception of covering spaces; easy connectedness arguments suffice instead.

In Section 6 we prove the Theorem. We exhibit a sequence of rational polyhedra $\mathfrak{t}_i$ and $\mathbb{Z}$-maps (projections) $\zeta_i \colon \mathfrak{t}_i \twoheadrightarrow \mathfrak{B}$ (see Figure 1). Each ζ_i is the dual of a unifier for ($\star$), because $\mathfrak{t}_i$ is the dual of a projective algebra by Cabrer's and Mundici's Lemma 3.8; and, by construction, these unifiers form an increasing sequence. Lemma 6.2 shows that the constructed sequence is in fact a strictly increasing, co-final sequence of unifiers for $\mathscr{E}$. The argument hinges on the lifting properties of a polyhedral model of the universal covering space of the circle $\mathbb{S}^1$; they are proved in Lemmas 5.1 and 5.3 of Section 5. Background notions on covering spaces are recalled in Subsection 2.5. The Theorem follows upon applying the duality theorem along with Ghilardi's Theorem 2.1.

In the final Section 7 we discuss further research.

2. Background and preliminaries.

2.1. MV-algebras.

Morphisms of MV-algebras are homomorphisms, i.e. functions preserving all operations. If θ is a congruence on an MV-algebra M, we write M/θ for the quotient algebra. The equivalence class of 0 with respect to θ, namely, $\ker\theta = \{x \in M \mid x \equiv 0 \,(\mathrm{mod}\,\theta)\}$, is an *ideal* of M. The equivalence class of 1 with respect to θ is a *filter* of M, and can be expressed as $\neg\ker\theta = \{\neg x \mid x \in \ker\theta\}$. Both $\ker\theta$ and $\neg\ker\theta$ uniquely determine θ. Ideals are characterised as the subsets J of M that include 0, are closed under $\oplus$, and are lower sets ($x \in J$ and $y \in M$ with $y \leqslant x$ implies $y \in J$). The usual homomorphism theorems can be proved for ideals; if M is an MV-algebra and J is an ideal of M, one writes M/J for the quotient algebra.

Since MV-algebras form a variety of algebras, free MV-algebras exist by Birkhoff's theorem [5, Theorem 10.10]. We write $\mathcal{F}_S$ for the MV-algebra freely generated by the set S. Recall that $\mathcal{F}_S$ is characterised by the following universal property: For every MV-algebra M and every set-theoretic function $\bar{h}\colon S \to M$, there exists a unique extension of $\bar{h}$ to a homomorphism of MV-algebras $h\colon \mathcal{F}_S \to M$. When S has finite cardinality $n \geqslant 0$, we write $\mathcal{F}_n$ in place of $\mathcal{F}_S$, and adopt the convention of identifying S with the set of "variables" $\{X_1, \ldots, X_n\}$.

As a special case of a universal-algebraic notion (see Subsection 2.3 below), an MV-algebra is *finitely presented* if it is (isomorphic to one) of the form $\mathcal{F}_n/J$, where $n \geqslant 0$ is an integer and J is a finitely generated ideal of $\mathcal{F}_n$. The latter condition means that there is a finite subset $F \subseteq M$ such that J is the intersection of all ideals of $\mathcal{F}_n$ containing J. An easy argument proves that the ideal J is finitely generated if and only it is principal (=singly generated) [10, 1.2.1].

2.2. Projective objects.

An object P in a category is called *projective* with respect to a class $\mathcal{E}$ of morphisms if for any $f\colon A \twoheadrightarrow B$ in $\mathcal{E}$ and any arrow $g\colon P \to B$, there exists an arrow $h\colon P \to A$ such that the following diagram commutes.

$$\begin{tikzcd} & A \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "g"'] & B \end{tikzcd}$$

The diagram expresses the so-called *projective lifting property* (applied to $\mathcal{E}$). The class $\mathcal{E}$ may consist of all epimorphisms, or of epimorphisms qualified in some manner; both regular and strong epimorphisms, for instance, have been used in the literature. In this paper, objects invariably are algebras in a variety, and the arrow $f\colon A \twoheadrightarrow B$ always is taken to be a surjection. It is well known that surjections in a variety are the same thing as regular epimorphisms, see e.g. [1, (vi) on p. 135].

An object A in a category is said to be a *retract* of an object B if there are arrows $s\colon A \hookrightarrow B$ and $r\colon B \twoheadrightarrow A$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[rr, "\mathbf{1}_A"] \arrow[dr, hook, "s"'] & & A \\
& B \arrow[ur, two heads, "r"'] &
\end{tikzcd}$$

(The arrow $\mathbf{1}_A$ is the identity morphism on A; we adopt this notation throughout for identity morphisms in a category.) When this is the case, r is called a *retraction* (of s) and s a *section* (of r). If the category in question is a variety, it follows at once that r is surjective, and s is injective.

One checks that on these definitions projective objects in any variety of algebras are stable under retractions, and they are precisely the retracts of free objects. In particular, free objects are projective.

2.3. Ghilardi's algebraic unification type.

Let us fix a variety $\mathbb{V}$ of algebras, and let us write $\mathcal{F}_I$ for the free object in $\mathbb{V}$ generated by a set I. Recall that an algebra A of $\mathbb{V}$ is *finitely presented* if it is a quotient of the form $A = \mathcal{F}_I/\theta$, with I finite and θ a finitely generated congruence. The elements of I are the *generators* of A, while any given subset of pairs $(s,t) \in \theta$ that generates the congruence θ is traditionally called a *set of relators* for A. In keeping with widespread usage we blur the distinction between finitely presented algebras and *finitely presentable* algebras, i.e. algebras isomorphic to some finitely presented algebra.

Following [14], by an *algebraic unification problem* we mean a finitely presented algebra A of $\mathbb{V}$. An *algebraic unifier* for A is a homomorphism $u\colon A \to P$ with P a finitely presented projective algebra[4] in $\mathbb{V}$; and A is *algebraically unifiable* if such an algebraic unifier exists.

Given another algebraic unifier $w\colon A \to Q$, we say that u *is more general than* w, written $w \preceq u$, if there is a homomorphism $g\colon P \to Q$ making the following diagram commute.

$$\begin{tikzcd}
& P \arrow[dd, "g"] \\
A \arrow[ur, "u"] \arrow[dr, "w"'] & \\
& Q
\end{tikzcd}$$

The relation $\preceq$ is a pre-order on the set $U(A)$ of algebraic unifiers for A. Let $\sim$ be the equivalence relation: $u \sim w$ if and only if both $u \preceq w$ and and $w \preceq u$ hold. The quotient set $\frac{U(A)}{\sim}$, whose elements are denoted $[u]$, is partially ordered by the relation: $[w] \leqslant [u]$ if and only if $w \preceq u$.

The *algebraic unification type* of an algebraically unifiable finitely presented algebra A in the variety $\mathbb{V}$ is now defined exactly as in the symbolic case (see the Introduction), using the partially ordered set $\left(\frac{U(A)}{\sim}, \leqslant\right)$ in place of $\left(\frac{U(\mathscr{E})}{\sim}, \leqslant_E^V\right)$. One also defines the *algebraic unification type* of the variety $\mathbb{V}$ in the same fashion.

Theorem 2.1. *Given an equational theory E with finite signature $\mathscr{F}$ over the set of variables $\mathscr{V} = \{X_1, X_2, \ldots\}$, let $\mathbb{V}_E$ be the variety of algebras axiomatised by E. Let I be a finite set, and consider the* (*symbolic*) *unification problem*

$$\mathscr{E} = \{\, (s_i\,,\, t_i) \;\mid\; i \in I \,\}\ ,$$

[4]It is an exercise to check that the finitely presented algebra P is projective in $\mathbb{V}_{\rm fp}$ — the category of finitely presented algebras of $\mathbb{V}$ and their homomorphisms — if and only if it is projective in $\mathbb{V}$. The expression 'finitely presented projective algebra' is therefore not ambiguous.

where $s_i, t_i \in \mathsf{Term}_{\mathscr{V}}(\mathscr{F})$ are terms, and $V \subseteq \mathscr{V}$ is the finite set of variables occurring in the terms s_i, t_i as i ranges in I. Let A be the algebra of $\mathbb{V}_E$ finitely presented by the relators $\mathscr{E}$ over the generators in V.

Then $\mathscr{E}$ is unifiable if and only if A is algebraically unifiable. Further, the partially ordered sets $\left(\frac{U(A)}{\sim}, \leqslant\right)$ of algebraic unifiers for A, and $\left(\frac{U(\mathscr{E})}{\sim}, \leqslant_E^V\right)$ of unifiers for $\mathscr{E}$, are isomorphic. In particular, the unification type of E and the algebraic unification type of $\mathbb{V}_E$ coincide.

Proof. This is Ghilardi's theorem [14, Theorem 4.1], stated in a form that is expedient for the sequel. $\square$

Remark 2.2. Although in this paper the category under investigation is a variety of finite signature, it would be misleading to suggest that Ghilardi's approach to unification is restricted to universal-algebraic contexts. Indeed, let us explicitly point out how one can define the unification type of an arbitrary locally small category C. (Recall that C is *locally small* if its hom-sets are sets rather than proper classes.) The following basic concept is due to the work of Gabriel and Ulmer [12]: an object A of C is *finitely presentable* if the covariant hom-functor $\mathrm{Hom}_{\mathsf{C}}(A, -)\colon \mathsf{C} \to \mathsf{Set}$ preserves filtered colimits. An object P in C is standardly defined to be (regular) projective if P has the projective lifting property of Subsection 2.2, applied to (regular) epimorphisms $f\colon A \twoheadrightarrow B$. With these notions available, one can define the unification type of C precisely as was done in the above for a variety, after Ghilardi's ideas. By a non-trivial result of Gabriel and Ulmer (see [1, Theorem 3.12] for an accessible proof), when C happens to be the category of algebras and homomorphisms of a variety $\mathbb{V}$, then the Gabriel-Ulmer finitely presentable objects of C coincide with the algebras in $\mathbb{V}$ that are isomorphic to some finitely presented algebra. Thus the categorial, algebraic, and symbolic unification types of a variety all coincide — provided 'projective' means 'regular projective', as in this paper. By contrast, such an abstraction of unification theory at the level of general categories is less viable for those approaches that use free algebras in place of projective ones: indeed, the requirement that C admits free objects (i.e. a forgetful functor to the category of sets, along with a left adjoint) is quite strong, and leaves us relatively close to algebraic categories; see e.g. [21, Chapter VI].

2.4. Rational polyhedral geometry.

Throughout this subsection, let us fix an integer $d \geqslant 0$ as the dimension of the real vector space $\mathbb{R}^d$. A *convex combination* of a finite set of vectors $v_1, \ldots, v_u \in \mathbb{R}^d$ is any vector of the form $\lambda_1 v_1 + \cdots + \lambda_u v_u$, for non-negative real numbers $\lambda_i \geqslant 0$ satisfying $\sum_{i=1}^{u} \lambda_i = 1$. If $S \subseteq \mathbb{R}^d$ is any subset, we let $\mathtt{conv}\, S$ denote the *convex hull* of S, i.e. the collection of all convex combinations of finite sets of vectors $v_1, \ldots, v_u \in S$. A *polytope* is any subset of $\mathbb{R}^d$ of the form $\mathtt{conv}\, S$, for some finite $S \subseteq \mathbb{R}^d$, and a (*compact*) *polyhedron* is a union of finitely many polytopes in $\mathbb{R}^d$. A polytope is *rational* if it may be written in the form $\mathtt{conv}\, S$ for some finite set $S \subseteq \mathbb{Q}^d \subseteq \mathbb{R}^d$ of vectors with rational coordinates. Similarly, a polyhedron is *rational* if it may be written as a union of finitely many rational polytopes. The *dimension* of a polyhedron P is the dimension of its *affine hull*, i.e. the affine subspace of $\mathbb{R}^d$ given by the intersection of all affine subspaces that contain P.

Recall that the vectors $v_0, v_1, \ldots, v_u \in \mathbb{R}^d$ are *affinely independent* if $\{v_1 - v_0, v_2 - v_0, \ldots, v_u - v_0\}$ is a linearly independent set. A polytope that may be written as $\sigma = \mathtt{conv}\, S$, for $S = \{v_0, v_1, \ldots, v_u\}$ a finite set of affinely independent vectors, is a (u-*dimensional*) *simplex*, or a u-*simplex* for short; S is then the (uniquely determined) set of *vertices* of σ. The simplex σ is *rational* if $S \subseteq \mathbb{Q}^d$. A (w-*dimensional*) *face* of σ is any simplex of the form $\mathtt{conv}\, S'$, for $S' \subseteq S$ a set of cardinality $w + 1$. A (*rational*) *simplicial complex* in $\mathbb{R}^d$ is a finite collection Σ of (rational) simplices in $\mathbb{R}^d$ such that any two simplices in Σ intersect in a common face. (This includes the case that the two simplices are disjoint: then, and only then, they intersect in $\emptyset$, their unique common -1-dimensional face.) The *dimension* of Σ is the maximum of the dimensions of its simplices. The simplices of Σ having dimension 0 are its *vertices*. The *support*, or *underlying polyhedron*, of Σ is $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$. It indeed is a (rational) polyhedron, by definition. Conversely, it is a basic fact that every (rational) polyhedron P is the support of some (rational) simplicial complex Σ; see e.g. [29, 2.11].

If $v \in \mathbb{Q}^d \subseteq \mathbb{R}^d$, there is a unique way to write out v in coordinates as

$$v = (\frac{p_1}{q_1}, \ldots, \frac{p_d}{q_d}) \ , \ p_i, q_i \in \mathbb{Q} \ , \ q_i > 0 \ , \ p_i \text{ and } q_i \text{ relatively prime for each } i = 1, \ldots, d \ .$$

Setting $q = \mathrm{lcm}\{q_1, q_2, \ldots, q_d\}$, the *homogeneous correspondent* of v is defined to be the integer vector

$$\tilde{v} = (\frac{qp_1}{q_1}, \ldots, \frac{qp_d}{q_d}, q) \in \mathbb{Z}^{d+1} .$$

The positive integer q is then called the *denominator* of v, written $\mathtt{den}\, v$. Clearly, $\mathtt{den}\, v = 1$ if and only if v has integers coordinates; following traditional terminology in the geometry of numbers, we call such v a *lattice point.* A rational u-dimensional simplex with vertices $v_0, \ldots, v_u$ is *unimodular* if the set $\{\tilde{v}_0, \ldots, \tilde{v}_u\}$ can be completed to a $\mathbb{Z}$-module basis of $\mathbb{Z}^{d+1}$; equivalently, if there is a $(d+1) \times (d+1)$ matrix with integer entries whose first u columns are $\tilde{v}_0, \ldots, \tilde{v}_u$, and whose determinant is ± 1. A simplicial complex is *unimodular* if each one of its simplices is unimodular.

Lemma 2.3. *Any rational polyhedron $P \subseteq \mathbb{R}^d$ is the support of a unimodular simplicial complex.*

Proof. This is proved in [24, 1.2] for $P \subseteq [0,1]^d$ as an application of Blichfeldt's lemma in the geometry of numbers: a bounded open set in $\mathbb{R}^d$ whose Lebesgue measure is > 1 contains a pair of distinct vectors whose difference is a lattice point; see e.g. [30, Lemma 1 on p. 13]. The same proof goes through for $P \subseteq \mathbb{R}^d$, *mutatis mutandis.* □

Throughout, the adjective 'linear' is to be understood as 'affine linear'. A function $f \colon \mathbb{R}^d \to \mathbb{R}$ is *piecewise linear* if it is continuous (with respect to the Euclidean topology on $\mathbb{R}^d$ and $\mathbb{R}$), and there is a finite set of linear functions $l_1, \ldots, l_u$ such that for each $x \in \mathbb{R}^d$ one has $f(x) = l_i$ for some choice of $i = 1, \ldots, u$. If, moreover, each l_i can be written as a linear polynomial with integer coefficients, then f is a *$\mathbb{Z}$-function* (or *$\mathbb{Z}$-map*). For an integer $d' \geqslant 0$, a function $\lambda = (\lambda_1, \ldots, \lambda_{d'}) \colon \mathbb{R}^d \to \mathbb{R}^{d'}$ is a *piecewise linear map* (respectively, a *$\mathbb{Z}$-map*) if each one of its scalar components $\lambda_j \colon \mathbb{R}^d \to \mathbb{R}$ is a piecewise linear function ($\mathbb{Z}$-function). We now define piecewise linear maps ($\mathbb{Z}$-maps) $A \to B$ for arbitrary subsets $A \subseteq \mathbb{R}^d$, $B \subseteq \mathbb{R}^{d'}$ as the restriction and co-restriction of piecewise linear maps ($\mathbb{Z}$-maps) $\mathbb{R}^d \to \mathbb{R}^{d'}$.

When the spaces at issue are rational polyhedra, a useful equivalent to the preceding definition of $\mathbb{Z}$-map is available.

Lemma 2.4. *Let $P \subseteq \mathbb{R}^d$ be a rational polyhedron, and let $f \colon P \to \mathbb{R}$ be a continuous function. Then the following are equivalent.*

1. *f is a $\mathbb{Z}$-function.*
2. *There exist finitely many linear polynomials with integer coefficients $l_1, \ldots, l_u \colon \mathbb{R}^d \to \mathbb{R}$ such that, for each $p \in P$, $f(p) = l_{i_p}(p)$ for some $i_p \in \{1, \ldots, u\}$.*

Proof. $(1 \Rightarrow 2)$ Trivial.

$(2 \Rightarrow 1)$ See the proof in [25, 3.1 and 3.2] for the case $P \subseteq [0,1]^d$, of which the case $P \subseteq \mathbb{R}^d$ is a variant. □

It is not hard to show that the composition of $\mathbb{Z}$-maps between rational polyhedra is again a $\mathbb{Z}$-map. A $\mathbb{Z}$-map $\lambda \colon A \to B$ between rational polyhedra $A \subseteq \mathbb{R}^d$ and $B \subseteq \mathbb{R}^{d'}$ is a *$\mathbb{Z}$-homeomorphism* if there exists a $\mathbb{Z}$-map $\lambda' \colon B \to A$ such that $\lambda \circ \lambda' = \mathbf{1}_B$ and $\lambda' \circ \lambda = \mathbf{1}_A$. In other words, a $\mathbb{Z}$-map is a $\mathbb{Z}$-homeomorphism if it is a homeomorphim whose inverse is a $\mathbb{Z}$-map, too. With these definitions, rational polyhedra and $\mathbb{Z}$-maps form a category.

Finally, we shall need the following lemma that relates the vanishing locus of $\mathbb{Z}$-functions to rational polyhedra.

Lemma 2.5. *For any subset $S \subseteq [0,1]^d$, the following are equivalent.*

1. *S is a rational polyhedron.*
2. *There is a $\mathbb{Z}$-function $\theta \colon [0,1]^d \to [0,1]$ vanishing precisely on S, that is, such that $\theta^{-1}(0) = S$.*

Proof. This is proved in [22, Proposition 5.1]. □

2.5. The universal cover of the circle.

Let us recall some standard notions from algebraic topology; we refer to [16] for details.

A *path* in a space X is a continuous map $f\colon [0,1] \to X$; the *endpoints* of f are $f(0)$ and $f(1)$. A space X is *path-connected* if for any $x_0, x_1 \in X$ there is a path in X with endpoints x_0, x_1. On the other hand, X is *locally path-connected* if each point has arbitrarily small open neighbourhoods that are path-connected; that is, for each $y \in X$ and each neighbourhood U of y there is a path-connected open neighbourhood of y contained in U. It is not hard to prove that polyhedra are locally path-connected (in fact, locally contractible by [16, Proposition A.1]), and therefore that a polyhedron is connected if and only if it is path-connected. We shall assume these statements in the following; none of them, of course, need to hold for more general spaces.

Two paths $f, g\colon [0,1] \to X$ with common endpoints x_0 and x_1 are *homotopic* if there is a *homotopy of paths* connecting them, i.e. a continuous function $h\colon [0,1]^2 \to X$ such that $h(s,1) = f(s)$ and $h(s,0) = g(s)$ for all $s \in [0,1]$, while $h(0,t) = x_0$, $h(1,t) = x_1$ for all $t \in [0,1]$. Homotopy of paths is an equivalence relation [16, Proposition 1.2]. A *loop* in X based at $x_0 \in X$ is a path $f\colon [0,1] \to X$ with $f(0) = f(1) = x_0$. The *fundamental group* of a space X at $x_0 \in X$, denoted $\pi_1(X, x_0)$, is the set of equivalence classes of loops in X based at $x_0 \in X$ under the equivalence relation of homotopy of paths. It indeed is a group upon associating to two paths f, g a third path $g \cdot f$ that traverses the union of the ranges of f and g in that order, at twice the original speed; see [16, Proposition 1.3]. When X is path-connected, the choice of basepoint is immaterial, and the fundamental group is denoted $\pi_1(X)$. A space X is *simply-connected* if it is path-connected, and $\pi_1(X)$ is the trivial (singleton) group. Also recall that the fundamental group is actually a (covariant) functor from the category of topological spaces with a distinguished basepoint and their basepoint-preserving continuous maps, to the category of groups and their homomorphisms. See [16, p. 34 and ff.].

A *covering space* [16, Section 1.3] of a topological space X is a space $\tilde{X}$ together with a surjective continuous map $p\colon \tilde{X} \twoheadrightarrow X$, called a *covering map*, such that there is a open covering $\{O_i\}$ of X, with i ranging in some index set I, satisfying the following condition: for each $i \in I$ the inverse image $p^{-1}(O_i)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically by p onto O_i.

If $p\colon \tilde{X} \to X$ is a covering map of the space X, and if Y is any space, a continuous map $f\colon Y \to X$ is said to *lift to* p (or, more informally, to $\tilde{X}$, when p is understood), if there is a continuous map $\tilde{f}\colon Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$. Any such $\tilde{f}$ is then called a *lift* of f. In the next lemma we recall two important properties of covering maps with respect to lifts that we will use in Section 5.

Lemma 2.6. *Given topological spaces X and $\tilde{X}$, suppose that $p\colon \tilde{X} \to X$ is a covering map. Further, let Y be a topological space, and let $f\colon Y \to X$ be a continuous map. Then the following hold.*

1. (Unique lifting property.) *Assume Y is connected. If $\tilde{f}, \tilde{f}'\colon Y \to X$ are two lifts of f that agree at one point of Y, then $\tilde{f} = \tilde{f}'$.*
2. (Lifting property of simply-connected polyhedra.) *If, additionally, Y is a simply-connected polyhedron, then a lift of f does exist. In fact, for any point $y \in Y$, and for any point $\tilde{x} \in \tilde{X}$ lying in the fibre over $f(y)$, i.e. such that $p(\tilde{x}) = f(y)$, there is a lift $\tilde{f}$ of f such that $\tilde{f}(y) = \tilde{x}$.*

Proof. 1. This is [16, Proposition 1.34].

2. As a special case of the general lifting criterion proved in [16, Proposition 1.33], we only need check that Y is both path-connected and locally path-connected. But since we are assuming that Y is simply-connected, it is path-connected by definition; and since it is a polyhedron, it is locally path-connected, too. □

Under appropriate conditions,[5] the space X has a simply-connected covering space called its *universal cover*. The universal cover of X is a covering space of every other path-connected covering space of X, and is essentially unique: indeed, it can be characterised by a universal property. See [16, Theorem 1.38].

[5] Namely, that X is path-connected, locally path-connected, and semilocally simply-connected; see again [16] for details.

Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle in the Euclidean plane $\mathbb{R}^2$, and let $\chi\colon \mathbb{R} \to \mathbb{S}^1$ be the continuous function given by

$$t \mapsto (\cos 2\pi t, \sin 2\pi t) \ .$$

Upon embedding $\mathbb{R}$ into $\mathbb{R}^3$ as a helix H via $t \mapsto (\cos 2\pi t, \sin 2\pi t, t)$, χ acts on H as the orthogonal projection onto $\mathbb{S}^1$ along the z-axis. The surjective map

$$\chi\colon \mathbb{R} \twoheadrightarrow \mathbb{S}^1$$

is the *universal covering map of the circle*, and $\mathbb{R}$ is the *universal cover* of $\mathbb{S}^1$; they will play a prominent rôle in the proof of the Theorem.

3. Duality for finitely presented MV-algebras and rational polyhedra.

3.1. Construction of the functor $\mathscr{M}$: Objects.

Let S be any subset of $\mathbb{R}^n$, for some integer $n \geqslant 0$. Let us write $\mathscr{M}(S)$ for the collection of all $\mathbb{Z}$-maps $S \to [0, 1]$. Regarding $[0, 1]$ as an MV-algebra with neutral element 0 under the operations $x \oplus y = \min\{x + y, 1\}$ and $\neg x = 1 - x$, for $x, y \in [0, 1]$, we can pull back an MV-algebraic structure on $\mathscr{M}(S)$ by defining operations pointwise. Specifically, let us define the functions $\mathbf{0}$, $\neg f$, $f \oplus g\colon S \to [0, 1]$, for each $f, g \in \mathscr{M}(S)$, by

$$\begin{aligned} \mathbf{0}(x) &= 0 \ , \\ (\neg f)(x) &= \neg(f(x)) = 1 - f(x) \ , \\ (f \oplus g)(x) &= f(x) \oplus g(y) = \min\{f(x) + g(x), 1\} \ , \end{aligned}$$

for $x, y \in S$. With these definitions,

$$\mathscr{M}(S) \equiv (\mathscr{M}(S), \oplus, \neg, \mathbf{0})$$

is an MV-algebra. In what follows, we shall always tacitly regard $\mathscr{M}(S)$ as an MV-algebra in this manner.

The MV-algebras of the form $\mathscr{M}(P)$, for P a rational polyhedron in $[0, 1]^n$, have a well-known characterisation. We begin with the case that P is a whole unit cube. Let us write $\xi_i\colon [0, 1]^n \twoheadrightarrow [0, 1]$, $i = 1, \ldots, n$, for the projection $(x_1, \ldots, x_n) \mapsto x_i$.

Lemma 3.1. *For any integer $n \geqslant 0$, the projection functions $\xi_1, \ldots, \xi_n$ generate $\mathscr{M}([0, 1]^n)$ freely.*

Proof. This is [10, 9.1.5]. □

Recall from Subsection 2.1 that we write $\mathcal{F}_n$, where $n \geqslant 0$ is an integer, for the MV-algebra freely generated by the set $\{X_1, \ldots, X_n\}$. If $s = s(X_1, \ldots, X_n)$ is an element of $\mathcal{F}_n$, in light of Lemma 3.1 we can write $s(\xi_1, \ldots, \xi_n)$ for the unique element of $\mathscr{M}([0, 1]^n)$ corresponding to s via the unique isomorphism that extends the assignment $X_i \mapsto \xi_i$, $i = 1, \ldots, n$. For general rational polyhedra in $[0, 1]^n$, we have:

Lemma 3.2. *For any integer $n \geqslant 0$ and for any congruence θ on the free MV-algebra $\mathcal{F}_n$, the following are equivalent.*

1. *The congruence θ is finitely generated.*
2. *The set $\ker\theta = \{g \in \mathcal{F}_n \mid g \equiv 0 \,(\mathrm{mod}\,\theta)\}$ is a principal ideal of $\mathcal{F}_n$.*
3. *There is a rational polyhedron $P \subseteq [0, 1]^n$ such that $\mathcal{F}_n/\theta \cong \mathscr{M}(P)$ via an isomorphism of MV-algebras that extends the map*
$$X_i/\theta \longmapsto \xi_i \restriction P \ , \ i = 1, \ldots, n \ ,$$
where $\xi_i \restriction P$ denotes the restriction of ξ_i to P.

4. The solution set of θ in $[0,1]^n$, namely,

$$\{x \in [0,1]^n \mid s(\xi_1(x), \ldots, \xi_n(x)) = t(\xi_1(x), \ldots, \xi_n(x)) \text{ for each } (s,t) \in \theta\} \ ,$$

is a rational polyhedron.

Proof. The equivalence (1 $\Leftrightarrow$ 2) is an elementary exercise on MV-algebras, see [10, 1.2.1 and 1.2.6]. The equivalence (2 $\Leftrightarrow$ 3) is [6, Lemma 2.5], or (in the language of unital Abelian lattice-ordered groups) [22, (i $\Leftrightarrow$ ii) in Corollary 5.2]. Finally, (3 $\Leftrightarrow$ 4) is a rephrasing of the equivalence (2 $\Leftrightarrow$ 3) that uses Lemma 3.1. $\square$

3.2. Construction of the functor $\mathscr{M}$: Arrows.

Let $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ be rational polyhedra, for integers $m, n \geqslant 0$. Let $\lambda\colon P \to Q$ be a $\mathbb{Z}$-map. Then there is an induced function

$$\mathscr{M}(\lambda)\colon \mathscr{M}(Q) \to \mathscr{M}(P)$$

given by

$$f \in \mathscr{M}(Q) \overset{\mathscr{M}(\lambda)}{\longmapsto} f \circ \lambda \in \mathscr{M}(P) \ .$$

Observe that the codomain of $\mathscr{M}(\lambda)$ indeed is $\mathscr{M}(P)$: the composition $f \circ \lambda$ of $\mathbb{Z}$-maps is again a $\mathbb{Z}$-map, and since the range of f is contained in $[0,1]$, so is the range of $f \circ \lambda$.

Lemma 3.3. *Let $\lambda\colon P \to Q$ be a $\mathbb{Z}$-map between rational polyhedra $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$, for integers $m, n \geqslant 0$. Then $\mathscr{M}(\lambda)\colon \mathscr{M}(Q) \to \mathscr{M}(P)$ is a homomorphism of MV-algebras.*

Proof. It is clear that $\mathscr{M}(\lambda)(\mathbf{0}) = \mathbf{0} \circ \lambda = \mathbf{0}$. Let $f, g \in \mathscr{M}(Q)$. For each $x \in P$, writing $\mathbf{1} = \neg\mathbf{0}\colon \mathscr{M}(Q) \to [0,1]$ for the function constantly equal to 1, we compute

$$\begin{aligned}(\mathscr{M}(\lambda)(\neg f))(x) &= ((\neg f) \circ \lambda)(x) = \neg f(\lambda(x)) = 1 - (f \circ \lambda)(x) = (\mathbf{1} - (\mathscr{M}(\lambda)(f)))(x) = \\ &= (\neg(\mathscr{M}(\lambda(f))))(x) \ .\end{aligned}$$

Further, let us write $+$ and min, respectively, for pointwise addition and minimum of pairs of real-valued functions. Then:

$$\begin{aligned}(\mathscr{M}(\lambda)(f \oplus g))(x) &= ((f \oplus g) \circ \lambda)(x) = (f \oplus g)(\lambda(x)) = \min\{f(\lambda(x)) + g(\lambda(x)) \ , \ 1\} = \\ &= (\min\{(f \circ \lambda) + (g \circ \lambda) \ , \ \mathbf{1}\})(x) = (\min\{(\mathscr{M}(\lambda)(f)) + (\mathscr{M}(\lambda)(g)) \ , \ \mathbf{1}\})(x) = \\ &= ((\mathscr{M}(\lambda)(f)) \oplus (\mathscr{M}(\lambda)(g)))(x) \ .\end{aligned}$$

This proves that $\mathscr{M}(\lambda)$ is a homomorphism of MV-algebras. $\square$

3.3. The duality theorem.

Recall [21, p. 14–15] that a functor between locally small categories is *faithful* (respectively, *full*) if it acts injectively (respectively, surjectively) on hom-sets, and it is *essentially surjective* if every object in the target category is isomorphic to some object in the range of the functor. A pair $S\colon \mathsf{C} \to \mathsf{D}$, $T\colon \mathsf{D} \to \mathsf{C}$ of functors is an *equivalence of categories* (and the categories C and D are *equivalent*) if $T \circ S$ and $S \circ T$ are naturally isomorphic to the identity functors on C and D, respectively [21, p. 93]. A well-known result [21, Theorem 1 on p. 93] is to the effect that a full, faithful, essentially surjective functor S has an adjoint T such that the pair S, T is an equivalence of categories.

A straightforward consequence of Lemmas 3.2 and 3.3 is that $\mathscr{M}$ is a functor from the category $\mathsf{P}_{\mathbb{Z}}$ of rational polyhedra, and the $\mathbb{Z}$-maps among them, to the opposite of the category $\mathsf{MV}_{\mathrm{fp}}$ of finitely presented MV-algebras, and their homomorphisms. Much more is true.

Theorem 3.4 (Duality theorem for finitely presented MV-algebras)**.** *The functor $\mathscr{M}\colon \mathsf{P}_{\mathbb{Z}} \to \mathsf{MV}_{\mathrm{fp}}^{\mathrm{op}}$ is full, faithful, and essentially surjective. Hence, the categories $\mathsf{MV}_{\mathrm{fp}}$ and $\mathsf{P}_{\mathbb{Z}}$ are dually equivalent.*

Proof. That $\mathscr{M}$ is essentially surjective follows at once from ($1 \Leftrightarrow 3$) in Lemma 3.2. To prove the other two properties, we first settle:

Claim 3.5. *Let $R \subseteq \mathbb{R}^n$ be a rational polyhedron, for some integer $n \geqslant 0$. Then there exist an integer $d \geqslant 0$, a rational polyhedron $P \subseteq [0,1]^d$, and a $\mathbb{Z}$-homeomorphism $\lambda\colon R \to P$.*

Proof of Claim 3.5. The rational polyhedron R has a unimodular triangulation Σ by Lemma 2.3. Let $v_1, \ldots, v_d$ be the vertices of Σ, and let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. Writing $\texttt{den}\, v_i$ as usual for the denominator of v_i, we set $\bar{e}_i = e_i / \texttt{den}\, v_i \in [0,1]^d$, for $i = 1, \ldots, d$, so that $\texttt{den}\, \bar{e}_i = \texttt{den}\, v_i$. Let us further set

$$I = \{\, \{i_1, \ldots, i_u\} \mid 1 \leqslant i_1 < \cdots < i_u \leqslant d \text{ and } \texttt{conv}\, \{v_{i_1}, \ldots v_{i_u}\} \in \Sigma \,\} \; .$$

Let Δ be the set of simplices in $[0,1]^d$ given by

$$\Delta = \{\, \texttt{conv}\, \{\bar{e}_{i_1}, \ldots, \bar{e}_{i_u}\} \mid \{i_1, \ldots, i_u\} \in I \,\} \; .$$

It is then easy to check that Δ is a simplicial complex (because Σ is), and that Δ is unimodular (because of our choice of the vertices of Δ as renormalised vectors of the standard basis). We need to show that $|\Sigma|$ and $|\Delta|$ are $\mathbb{Z}$-homeomorphic. For this, let us consider first the function

$$\bar{\lambda}\colon \{v_1, \ldots, v_d\} \to \{\bar{e}_1, \ldots, \bar{e}_d\}$$

given by

$$v_i \stackrel{\bar{\lambda}}{\longmapsto} \bar{e}_i \;\; , \quad \text{for each } i = 1, \ldots, d.$$

By our definition of Δ, $\bar{\lambda}$ induces the inclusion-preserving bijection from Σ to Δ given by

$$\texttt{conv}\, \{v_{i_1}, \ldots, v_{i_u}\} \in \Sigma \mapsto \texttt{conv}\, \{\bar{e}_{i_1}, \ldots, \bar{e}_{i_u}\} \in \Delta \, .$$

Therefore [29, Exercise on p. 17, and 2.18], $\bar{\lambda}$ also induces by linear extension a unique piecewise linear homeomorphism

$$\lambda\colon |\Sigma| \to |\Delta| \; ,$$

namely, the unique such map that is linear over each simplex of Σ, and agrees with $\bar{\lambda}$ on the vertices of Σ. Using Lemma 2.4, a simple computation in linear algebra shows that our renormalisation of each e_i to $\bar{e}_i$ guarantees that λ is in fact a $\mathbb{Z}$-map. By the same token, the inverse function $\bar{\lambda}^{-1}$ induces an inclusion-preserving bijection from Δ to Σ, and then by linear extension a unique $\mathbb{Z}$-map

$$\lambda'\colon |\Delta| \to |\Sigma| \; .$$

By construction, $\lambda' \circ \lambda$ and $\lambda \circ \lambda'$ are the identity maps on $|\Sigma|$ and $|\Delta|$, respectively, so that λ is a $\mathbb{Z}$-homeomorphism. Taking $P = |\Delta|$ settles the claim. ☐

To check that $\mathscr{M}$ is faithful, let $\lambda_1, \lambda_2\colon P \to Q$ be $\mathbb{Z}$-maps between rational polyhedra $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$, for integers $m, n \geqslant 0$. By Lemma 3.3, $\mathscr{M}(\lambda_1), \mathscr{M}(\lambda_2)\colon \mathscr{M}(Q) \to \mathscr{M}(P)$ are homomorphisms of MV-algebras. We first show that it is enough to prove faithfulness when Q lies in a unit cube. For this, using Claim 3.5 we choose a $\mathbb{Z}$-homeomorphism

$$\gamma\colon Q \to Q' \, , \tag{1}$$

where $Q' \subseteq [0,1]^{n'}$ is a rational polyhedron, for some integer $n' \geqslant 0$. Then by composition we get $\mathbb{Z}$-maps

$$\lambda_1' = \gamma \circ \lambda_1 \colon P \to Q'$$
$$\lambda_2' = \gamma \circ \lambda_2 \colon P \to Q' \, .$$

But now $\lambda_1' = \lambda_2'$ if and only if $\gamma \circ \lambda_1 = \gamma \circ \lambda_2$ if and only if $\gamma^{-1} \circ (\gamma \circ \lambda_1) = \gamma^{-1} \circ (\gamma \circ \lambda_2)$ if and only if $\lambda_1 = \lambda_2$, as was to be shown. Hence we shall safely assume $Q \subseteq [0,1]^n$.

Suppose there is $p \in P$ such that $\lambda_1(p) = (x_1, \dots, x_n) \neq (x'_1, \dots, x'_n) = \lambda_2(p)$; without loss of generality, say $x_1 \neq x'_1$. Consider the projection $\xi^*_1 \colon Q \to [0,1]$ onto the first coordinate. Then $\xi^*_1 \in \mathscr{M}(Q)$ because $Q \subseteq [0,1]^n$. We have

$$(\mathscr{M}(\lambda_1)(\xi^*_1))(p) \;=\; \xi^*_1(\lambda_1(p)) = x_1 \ ,$$

whereas

$$(\mathscr{M}(\lambda_2)(\xi^*_1))(p) \;=\; \xi^*_1(\lambda_2(p)) = x'_1 \ .$$

Hence $\mathscr{M}(\lambda_1)$ disagrees with $\mathscr{M}(\lambda_2)$ at ξ^*_1 because $x_1 \neq x'_1$ by assumption. This shows that $\mathscr{M}$ is faithful.

To show that $\mathscr{M}$ is full, we consider a homomorphism $h \colon \mathscr{M}(Q) \to \mathscr{M}(P)$, with $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ rational polyhedra, and prove that there exists a $\mathbb{Z}$-map $\lambda \colon P \to Q$ such that $\mathscr{M}(\lambda) = h$. We first perform a reduction to the case when Q lies in a unit cube. Let γ be the $\mathbb{Z}$-homeomorphism in (1), and set

$$h' = h \circ \mathscr{M}(\gamma) \colon \mathscr{M}(Q') \to \mathscr{M}(P)\,.$$

Assume further that there is a $\mathbb{Z}$-map $\lambda' \colon P \to Q'$ such that $\mathscr{M}(\lambda') = h'$. Then the $\mathbb{Z}$-map $\lambda = \gamma^{-1} \circ \lambda' \colon P \to Q$ satisfies

$$\mathscr{M}(\lambda) = \mathscr{M}(\lambda') \circ \mathscr{M}(\gamma^{-1}) = h' \circ \mathscr{M}(\gamma^{-1}) = h \circ (\mathscr{M}(\gamma) \circ \mathscr{M}(\gamma^{-1})) = h \circ (\mathscr{M}(\gamma^{-1} \circ \gamma)) = h\,.$$

Hence we may safely assume that $Q \subseteq [0,1]^n$.

Let $\xi^*_1, \dots, \xi^*_n \colon Q \to [0,1]$ be the projection functions onto the i^{th} coordinate, $i = 1, \dots, n$. Then $\Xi^* = \{\xi^*_i\}_{i=1}^n \subseteq \mathscr{M}(Q)$, because $Q \subseteq [0,1]^n$. Further, Ξ^* is a generating set of $\mathscr{M}(Q)$. Indeed, the projection functions $\xi_1, \dots, \xi_n \colon [0,1]^n \to [0,1]$ form a generating set of the free MV-algebra $\mathscr{M}([0,1]^n)$ by Lemma 3.1. The restriction map to Q

$$\cdot^* \colon f \in \mathscr{M}([0,1]^n) \longmapsto f^* \in \mathscr{M}(Q)$$

is an onto homomorphism of MV-algebras that takes ξ_i to ξ^*_i, by ($1 \Leftrightarrow 3$) in Lemma 3.2. It coincides with the unique extension of the assignment

$$\xi_i \mapsto \xi^*_i \ , \ i = 1, \dots, n\,,$$

whose existence is guaranteed by the universal property of $\mathscr{M}([0,1]^n)$. Since $\cdot^* \colon \mathscr{M}([0,1]^n) \to \mathscr{M}(Q)$ is onto, it follows that Ξ^* generates $\mathscr{M}(Q)$.

Now consider the elements

$$\lambda_i = h(\xi^*_i) \in \mathscr{M}(P) \ , \ i = 1, \dots, n\,, \tag{2}$$

and define the function

$$\lambda(p) = (\lambda_1(p), \dots, \lambda_n(p)) \ , \ \text{for } p \in P\,. \tag{3}$$

By definition, λ is a $\mathbb{Z}$-map with domain P and range contained in $[0,1]^n$. Let us first prove that the range of λ actually is contained Q.

By Lemma 2.5, there is an element $\theta \in \mathscr{M}([0,1]^n)$ that vanishes precisely on Q, i.e. $\theta^{-1}(0) = Q$. Therefore, it is sufficient to prove that $\theta(\lambda(p)) = 0$ for every $p \in P$. Since $\xi_1, \dots, \xi_n$ generate $\mathscr{M}([0,1]^n)$, there is a term $\tau(X_1, \dots, X_n)$ in the language of MV-algebras such that

$$\theta = \tau(\xi_1, \dots, \xi_n)\,. \tag{4}$$

Since the restriction $\theta^* = \tau(\xi^*_1, \dots, \xi^*_n)$ of θ to Q is the element $\mathbf{0}$ of $\mathscr{M}(P)$, and h is a homomorphim, we have

$$h(\theta^*) = 0\,, \tag{5}$$

i.e. $h(\theta^*)$ is the function identically zero on P. Because h is a homomorphism, from (4–5) we infer

$$h(\theta^*) = \tau(h(\xi_1^*), \dots, h(\xi_n^*)) = 0\,. \tag{6}$$

Because the operations of $\mathscr{M}(P)$ are defined pointwise, and in light of (2), the evaluation of (6) at every $p \in P$ yields

$$\begin{aligned}[\tau(h(\xi_1^*), \dots, h(\xi_n^*))](p) &= \tau([h(\xi_1^*)](p), \dots, [h(\xi_n^*)](p)) \\ &= \tau(\lambda_1(p), \dots, \lambda_n(p)) = 0\end{aligned} \tag{7}$$

Using (3) and (4), the latter equality (7) reads

$$\theta(\lambda(p)) = 0\,,$$

as was to be shown.

Having shown that the range of λ is contained in Q, we can regard λ as a $\mathbb{Z}$-map $\lambda\colon P \to Q$. It remains to show that $\mathscr{M}(\lambda) = h$. It suffices to check that the two homomorphisms $\mathscr{M}(\lambda)$ and h agree at the generating set Ξ^* of $\mathscr{M}(Q)$. And indeed,

$$\begin{aligned}(\mathscr{M}(\lambda))(\xi_i^*) &= \xi_i^* \circ \lambda && \text{(by the definition of } \mathscr{M}\text{)},\\ &= \lambda_i && \text{(by (3) and the definition of } \xi_i^*\text{)},\\ &= h(\xi_i^*) && \text{(by (2))}.\end{aligned}$$

This completes the proof. □

3.4. Projective MV-algebras.

Corollary 3.6. *For any MV-algebra A, the following are equivalent.*

1. *A is finitely presented and projective.*
2. *A is finitely generated and projective.*
3. *Whenever $P \subseteq [0,1]^n$, for some integer $n \geqslant 1$, is a rational polyhedron with $\mathscr{M}(P) \cong A$, P is a retract of $[0,1]^n$ by $\mathbb{Z}$-maps.*

In particular, the full subcategory of $\mathsf{MV}_{\mathrm{fp}}$ whose objects are projective MV-algebras is dual to the full subcategory of $\mathsf{P}_{\mathbb{Z}}$ whose objects are retracts by $\mathbb{Z}$-maps of finite-dimensional unit cubes.

Proof. ($1 \Rightarrow 2$) Trivial.

($2 \Rightarrow 3$) Since $A \cong \mathscr{M}(P)$ is projective, it is a retract of any free finitely generated object of which it is a quotient. Since the dual of the inclusion map $\iota\colon P \hookrightarrow [0,1]^n$ is the quotient map $r\colon \mathcal{F}_n \twoheadrightarrow \mathscr{M}(P)$ by Lemma 3.1, it follows that r has a section $s\colon \mathscr{M}(P) \hookrightarrow \mathcal{F}_n$. Now $\mathscr{M}(s)\colon [0,1]^n \twoheadrightarrow P$ is the desired retraction by a $\mathbb{Z}$-map.

($3 \Rightarrow 1$) An exercise in polyhedral geometry shows that a retract of $[0,1]^n$ by $\mathbb{Z}$-maps is a rational polyhedron, so that $A \cong \mathscr{M}(P)$ is finitely presented by the duality theorem. To see that $\mathscr{M}(P)$ is projective, let $r\colon [0,1]^n \twoheadrightarrow \mathscr{M}(P)$ be a $\mathbb{Z}$-map that is a retraction, and let $s\colon \mathscr{M}(P) \hookrightarrow [0,1]^n$ be a $\mathbb{Z}$-map that is a section of r. Again by the duality theorem and Lemma 3.1, $\mathscr{M}(s)$ is a retraction of $\mathcal{F}_n$ onto $\mathscr{M}(P) \cong A$. As remarked in Subsection 2.2, in any variety retracts of projective objects are projective, and free objects are trivially projective; hence A is projective.

The last assertion follows at once from the duality theorem using ($1 \Leftrightarrow 3$). □

Remark 3.7. Corollary 3.6 is [6, Theorem 1.2].

We recall next a result on retracts by $\mathbb{Z}$-maps that will be applied in the proof of the Theorem. A unimodular simplicial complex Σ of dimension $\leqslant 1$ is *strongly regular* if any two vertices of Σ that span a 1-simplex have relatively prime denominators. Like dimension, strong regularity is a property of the underlying rational polyhedron $P = |\Sigma|$, and not of the complex Σ. In other words, suppose $P = |\Delta| = |\nabla|$ for two unimodular simplicial complexes Δ, ∇. Then Δ has dimension $\leqslant d$ if and only if ∇ has dimension $\leqslant d$, by the classical invariance of dimension; and Δ is strongly regular if and only if ∇ is, by [7, Lemma 2.5]. Accordingly, we shall speak of strongly regular rational polyhedra having dimension $\leqslant 1$.

Lemma 3.8. *For some integer $n \geqslant 1$, let P be a rational polyhedron in $[0,1]^n$ having dimension $\leqslant 1$. Suppose P is simply-connected, strongly regular, and contains a lattice point. Then P is a retract of $[0,1]^n$ by $\mathbb{Z}$-maps.*

Proof. This is proved in [7, Corollary 4.4]. □

4. The one-variable case.

Theorem 4.1. *Let $s_i(X_1)$ and $t_i(X_1)$ be terms in the language of MV-algebras built from the single variable X_1, for i ranging in some finite index set I. Then, if the unification problem $\mathscr{E} = \{(s_i(X_1), t_i(X_1)) \mid i \in I\}$ is unifiable, it admits either one mgu, or two maximally general unifiers that are more general than any other unifier for $\mathscr{E}$. Further, each one of these cases obtains for some choice of $\mathscr{E}$.*

The proof will require three lemmas. By a *rational interval* (or *segment*) in $[0,1]$ we mean a closed interval $[a,b] \subseteq [0,1]$ with $a, b \in \mathbb{Q}$. We regard singletons $\{a\}$ with $a \in [0,1] \cap \mathbb{Q}$ as degenerate cases ($a = b$) of rational segments.

Lemma 4.2. *The retracts of $[0,1]$ by $\mathbb{Z}$-maps are precisely the rational intervals in $[0,1]$ that contain a lattice point.*

Proof. Consider an interval $[0,q]$, with $q \in [0,1] \cap \mathbb{Q}$. The interval $[q,1]$ is the support of a unimodular triangulation Σ by Lemma 2.3; let $q' \in (q,1]$ be the unique vertex of Σ adjacent to q. Let now $r\colon [0,1] \to [0,1]$ be the piecewise linear function that coincides with the identity over $[0,q]$, with the zero function on $[q',1]$, and that is linear over the interval $[q,q']$. A simple computation shows that r is a $\mathbb{Z}$-map because of our choice of q'. It is clear that r retracts $[0,1]$ onto $[0,q]$. Conversely, let $r\colon [0,1] \twoheadrightarrow Q$ be any $\mathbb{Z}$-map that retracts $[0,1]$ onto $Q \subseteq [0,1]$. Then Q is the range of r. Since r is continuous, Q is compact and connected (because the domain $[0,1]$ of r is both), hence a closed interval. Since, moreover, r is a $\mathbb{Z}$-map, the maximum and minimum values it attains are rational – so that Q is a rational interval — and R contains at least one lattice point — the image under r of 0 and 1, which is necessarily a lattice point by the definition of $\mathbb{Z}$-map. □

Lemma 4.3. *Let P be a rational polyhedron in $[0,1]$. If $0 \in P$, there is a unique inclusion-maximal rational segment in $[0,1]$ that contains 0 and is contained in P. Similarly, if $1 \in P$, there is a unique inclusion-maximal rational segment in $[0,1]$ that contains 1 and is contained in P.*

Proof. By symmetry, we only argue for the case $0 \in P$. Let Σ be a rational triangulation of P. Then each simplex of Σ has dimension $\leqslant 1$. If $0 \in P$, then 0 must be a vertex of Σ, because it is not a convex combination of any subset of $[0,1]$ that omits 0. If 0 is not contained in any 1-simplex of Σ, then 0 is an isolated point of P, and $\{0\}$ is then the unique inclusion-maximal (degenerate) rational segment in $[0,1]$ that contains 0 and is contained in P. Otherwise, let $\sigma_0, \ldots, \sigma_l$ be the finite list of 1-simplices of Σ such that (i) $0 \in \sigma_0$, (ii) $\sigma_i \cap \sigma_{i+1} \neq \emptyset$ for $i = 0, \ldots, l-1$, and (iii) $\tau \cap \sigma_i = \emptyset$ for each $i = 0, \ldots, l$ and for each 1-simplex $\tau \notin \{\sigma_i\}_{i=0}^{l}$ of Σ. Such a list necessarily exists because Σ is a finite set. It is then clear that $S = \bigcup_{i=0}^{l} \sigma_i$ is a rational segment, because each σ_i has rational vertices, and that S is the unique inclusion-maximal rational segment in $[0,1]$ that contains 0 and is contained in P. □

Let P be a rational polyhedron in $[0,1]$. The 0-*component* of P, denoted $C_0(P)$, is defined to be $\emptyset$, if $0 \notin P$, and to be the unique inclusion-maximal rational segment of $[0,1]$ that contains 0 and is contained in P, otherwise. The existence and uniqueness of the latter segment is guaranteed by Lemma 4.3. Similarly, the 1-*component* of P, denoted $C_1(P)$, is defined to be $\emptyset$, if $1 \notin P$, and to be the unique inclusion-maximal rational segment of $[0,1]$ that contains 1 and is contained in P, otherwise. We say that P has *character*

- $\triangleright$ 0, if $C_0(P) = C_1(P) = \emptyset$;
- $\triangleright$ 1, if exactly one of $C_0(P)$ and $C_1(P)$ is non-empty;
- $\triangleright$ 2, if $C_0(P) \neq \emptyset \neq C_1(P)$, and $C_0(P) \neq C_1(P)$; and
- $\triangleright$ 3, if $C_0(P) = C_1(P) = [0,1]$.

Lemma 4.4. *For any rational polyhedron P in $[0,1]$, let $\chi \in \{0,1,2,3\}$ be the character of P. For some integer $d \geqslant 0$, let further $R \subseteq [0,1]^d$ be a rational polyhedron that is a retract of $[0,1]^n$ by $\mathbb{Z}$-maps. Then exactly one of the following cases obtains.*

(*a*) $\chi = 0$. *Then there is no $\mathbb{Z}$-map $R \to P$.*

(*b*) $\chi = 1$. *Then every $\mathbb{Z}$-map $R \to P$ factors via a $\mathbb{Z}$-map either through the injection $C_0(P) \hookrightarrow P$, or through the injection $C_1(P) \hookrightarrow P$, according as $C_0(P) \neq \emptyset$ or $C_1(P) \neq \emptyset$.*

(*c*) $\chi = 2$. *Then every $\mathbb{Z}$-map $R \to P$ factors via a $\mathbb{Z}$-map either through the injection $C_0(P) \hookrightarrow P$, or through the injection $C_1(P) \hookrightarrow P$, but not through both.*

(*d*) $\chi = 3$. *Then every $\mathbb{Z}$-map $R \to P = [0,1]$ factors through the identity map $\mathbf{1}_{[0,1]} \colon [0,1] \to [0,1]$.*

Proof. It is clear that each one of the above mutually exclusive cases (a–d) obtains for some choice of P. If (a) does, then $P \cap \{0,1\} = \emptyset$. A trivial computation shows that any $\mathbb{Z}$-map $f \colon R \to R' \subseteq [0,1]^{d'}$ is such that, for each $p \in R \cap \mathbb{Q}^d$, $\mathtt{den}\, f(p)$ divides $\mathtt{den}\, p$. In particular, f carries lattice points to lattice points. Thus, since R is a retract of $[0,1]^d$ by $\mathbb{Z}$-maps, it must contain a lattice point. But then there can be no $\mathbb{Z}$-map $R \to P$ if $P \cap \{0,1\} = \emptyset$, as P contains no lattice point in this case.

As to the other cases, Lemma 4.2 shows that $C_0(P)$ and $C_1(P)$ are retracts of $[0,1]$ by $\mathbb{Z}$-maps whenever they are non-empty. Say that in case (b) the 0-component $C_0(P)$ of P is non-empty, whereas $C_1(P) = \emptyset$. Then $C_0(P) = [0,q] \subseteq [0,1]$ for a unique rational number $0 \leqslant q < 1$, by Lemma 4.3. Next suppose $u \colon R \to P$ is a $\mathbb{Z}$-map. Since R is a retract of $[0,1]^d$ by $\mathbb{Z}$-maps, it is connected and contains a lattice point. It follows at once that the range of u is an interval that contains a lattice point. By our assumptions $\chi = 1$ and $[0,q] = C_0(P) \neq \emptyset$, the range of u is then contained in $C_0(P)$. Let now $g \colon R \to C_0(P)$ denote the co-restriction of $u \colon R \to P$ to $C_0(P) \subseteq P$. Then obviously $u = \iota_q \circ g$, as was to be shown.

Case (c) is proved by the same argument used for (b). The fact that no $\mathbb{Z}$-map $u \colon R \to P$ can factor through both injections $C_0(P), C_1(P) \hookrightarrow P$ follows upon noting that we must have $C_0(P) \cap C_1(P) = \emptyset$, and that therefore (by the connectedness of R) any such u must have range entirely included either in $C_0(P)$ or in $C_1(P)$. Case (d) is trivial. $\square$

End of Proof of Theorem 4.1. Let $\mathcal{F}_1$ be the MV-algebra freely generated by the element X_1. Let θ be the congruence relation on $\mathcal{F}_1$ generated by the subset of pairs $\mathscr{E}$. Since θ is finitely generated by construction, $A = \mathcal{F}_1/\theta$ is finitely presented. By Lemma 3.1 together with the duality theorem, the dual of the quotient map $\mathcal{F}_1 \twoheadrightarrow A$ is the inclusion map $P \hookrightarrow [0,1]$, for P a rational polyhedron in $[0,1]$ such that $\mathscr{M}(P) \cong A$. Let $\chi \in \{0,1,2,3\}$ be the character of P, let $C_i(P)$ be the i-component of P, and let $U_i = \mathscr{M}(C_i(P))$, $i = 0,1$. Each inclusion map $\iota_i \colon C_i(P) \hookrightarrow P$ dualises to a quotient map $q_i \colon A \twoheadrightarrow U_i$, $i = 0,1$. (For the sake of clarity, let us explicitly observe that if $C_i(P) = \emptyset$ then $U_i = \mathscr{M}(\emptyset)$ is the trivial one-element MV-algebra, the terminal object in the category; it is obviously finitely presentable, e.g. by $1 = 0$, but not projective.) Now let B be any finitely presented projective MV-algebra. By the duality theorem, let $R \subseteq \mathbb{R}^{d'}$ be a rational polyhedron such that $\mathscr{M}(R) \cong B$, for some integer $d' \geqslant 0$. By Claim 3.5 we may safely assume that $R \subseteq [0,1]^d$, for some integer $d \geqslant 0$. Then, by Corollary 3.6, R is a retract of $[0,1]^d$.

If now $\chi = 0$, the dual of (a) in Lemma 4.4 states that there is no homomorphism $A \to B$, so that A is not algebraically unifiable. If $\chi = 1$, say $C_0(P) \neq \emptyset$, so that U_0 is projective and $q_0 \colon A \twoheadrightarrow U_0$ is an algebraic unifier. Then the dual of (b) in Lemma 4.4 states that q_0 is the most general algebraic unifier for A. If $\chi = 2$, both U_0 and U_1 are projective, so the dual of (c) in Lemma 4.4 states that both q_0 and q_1 are distinct maximally general algebraic unifiers for B, and that any other other algebraic unifier for B is below either q_0 or q_1. Finally, if $\chi = 3$ so that $P = [0, 1]$, then (d) in Lemma 4.4 states that the identity $\mathbf{1}_A \colon A \to A$ is the most general algebraic unifier for A.

The proof is completed by applying Theorem 2.1 to the preceding analysis by cases according to the value of χ. □

5. Lifts of $\mathbb{Z}$-maps.

Let $\mathfrak{B}$ be the rational polyhedron consisting of the boundary of the unit square in $\mathbb{R}^2$. In symbols, if $v_1 = (1, 0), v_2 = (1, 1), v_3 = (0, 1), v_4 = (0, 0)$, then

$$\mathfrak{B} = \mathtt{conv}\,\{v_1, v_2\} \cup \mathtt{conv}\,\{v_2, v_3\} \cup \mathtt{conv}\,\{v_3, v_4\} \cup \mathtt{conv}\,\{v_4, v_1\} \ .$$

If $X, Y \subseteq \mathbb{R}^d$ are arbitrary subsets, for $d \geqslant 0$ an integer, their *Minkowski sum* is the set

$$X \dot{+} Y = \{v \in \mathbb{R}^d \mid x + y = v \text{ for some } x \in X, y \in Y\} \ .$$

When $X = \{x\}$ is a singleton, we write $x \dot{+} Y$ instead of $\{x\} \dot{+} Y$. In this case, $x \dot{+} Y$ is just the translation of Y by x.

Define the polyhedron $\mathtt{t}_1^+ \subseteq \mathbb{R}^3$ as

$$\mathtt{t}_1^+ = \mathtt{conv}\,\{(1,0,0),(1,1,0)\} \cup \mathtt{conv}\,\{(1,1,0),(0,1,0)\} \cup \mathtt{conv}\,\{(0,1,0),(0,0,0)\} \cup \mathtt{conv}\,\{\,(0,0,0),(1,0,1)\}\ ,$$

and the polyhedron $\mathtt{t}_1^- \subseteq \mathbb{R}^3$ as

$$\mathtt{t}_1^- = (0,0,-1) \dot{+} \mathtt{t}_1^+ \ .$$

Further, for each integer $i \geqslant 2$, set

$$\begin{aligned}
\mathtt{t}_i^+ &= \mathtt{t}_1^+ \ \cup\ (\,(0,0,1) \dot{+} \mathtt{t}_1^+\,) \ \cup\ \cdots \ \cup\ (\,(0,0,i-1) \dot{+} \mathtt{t}_1^+\,)\ ,\\
\mathtt{t}_i^- &= \mathtt{t}_1^- \ \cup\ (\,(0,0,-1) \dot{+} \mathtt{t}_1^-\,) \ \cup\ \cdots \ \cup\ (\,(0,0,-(i-1)) \dot{+} \mathtt{t}_1^-\,)\ ,\\
\mathtt{t}_i &= \mathtt{t}_i^+ \ \cup\ \mathtt{t}_i^- \ .
\end{aligned}$$

Finally, define

$$\mathtt{t}_\infty = \bigcup_{i \geqslant 1} \mathtt{t}_i \ .$$

See Figure 1.

For each integer $i \geqslant 1$, we have an inclusion $\mathbb{Z}$-map

$$\eta_i \colon \mathtt{t}_i \hookrightarrow \mathtt{t}_{i+1}\ ,$$

and a further (onto) projection $\mathbb{Z}$-map

$$\zeta_i \colon \mathtt{t}_i \twoheadrightarrow \mathfrak{B}$$

given by

$$(x, y, z) \in \mathtt{t}_i \ \overset{\zeta_i}{\longmapsto}\ (x, y) \in \mathfrak{B}\ . \tag{8}$$

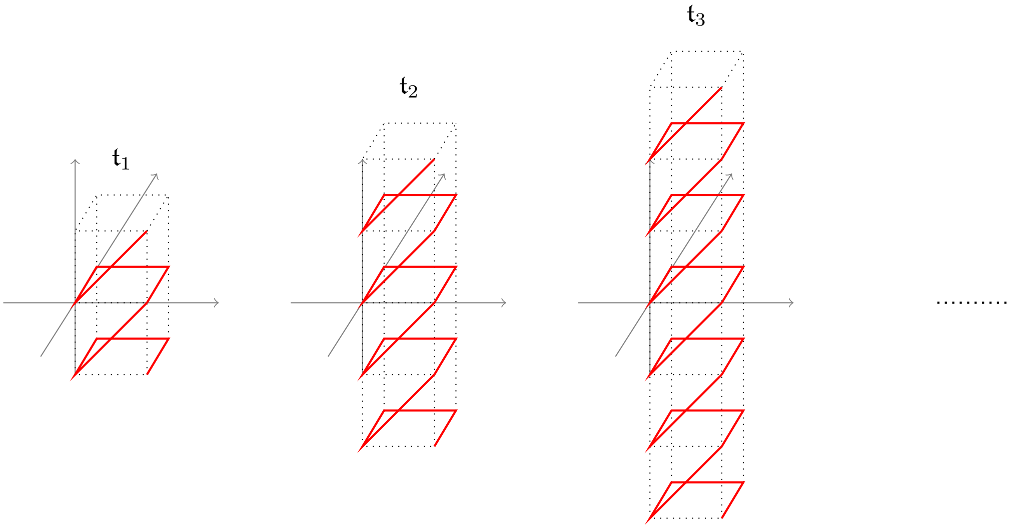

Figure 1: The rational polyhedra $\mathfrak{t}_i$ used in the proof of the Theorem.

We also have a (continuous) projection map

$$\zeta \colon \mathfrak{t}_\infty \twoheadrightarrow \mathfrak{B}$$

defined by the obvious analogue of (8); the restriction of ζ to $\mathfrak{t}_i$ is precisely ζ_i, for each integer $i \geqslant 1$. We shall retain the notation above for the remaining part of this paper.

It is clear by construction that ζ is a covering map, so that $\mathfrak{t}_\infty$ is a covering space of $\mathfrak{B}$. It is therefore easy to prove the following lemma by a lifting argument.

Lemma 5.1. *Let $i, j \geqslant 1$ be integers. If there is a continuous map $f \colon \mathfrak{t}_i \to \mathfrak{t}_j$ such that $\zeta_i = \zeta_j \circ f$, then f is injective.*

Proof. Evidently, both $\mathfrak{t}_i$ and $\mathfrak{t}_j$ are homeomorphic to a compact interval in $\mathbb{R}$, so that both are simply-connected because every convex subset of $\mathbb{R}^n$ is [16, Example 1.4]. By 2 in Lemma 2.6, there is a continuous map $\tilde{\zeta}_j \colon \mathfrak{t}_j \to \mathfrak{t}_\infty$ such that $\zeta_j = \zeta \circ \tilde{\zeta}_j$. Moreover, we *claim* that we can choose $\tilde{\zeta}_j$ as the unique such lift of ζ_j that satisfies $\tilde{\zeta}_j(f(1,0,0)) = (1,0,0) \in \mathfrak{t}_\infty$. For this, it suffices to show that $(1,0,0) \in \mathfrak{t}_\infty$ lies in the fibre over $\zeta_j(f(1,0,0))$. That is, we need to check that $\zeta(1,0,0) = \zeta_j(f(1,0,0))$, and since $\zeta(1,0,0) = (1,0)$, this amounts to checking that $\zeta_j(f(1,0,0)) = (1,0)$. But since $\zeta_i = \zeta_j \circ f$ by hypothesis, we have $\zeta_j(f(1,0,0)) = \zeta_i(1,0,0) = (1,0)$, and the *claim* is settled.

From $\zeta_j = \zeta \circ \tilde{\zeta}_j$ and the hypothesis $\zeta_i = \zeta_j \circ f$ we obtain

$$\zeta_i = \zeta \circ \tilde{\zeta}_j \circ f \ . \tag{9}$$

Equation (9) states that $\tilde{\zeta}_j \circ f$ is a lift of ζ_i. Now observe that the inclusion map $\tilde{\zeta}_i \colon \mathfrak{t}_i \to \mathfrak{t}_\infty$ also is a lift of ζ_i. We further have

$$\tilde{\zeta}_j(f(1,0,0)) = (1,0,0) = \tilde{\zeta}_i(1,0,0) \tag{10}$$

because of our choice of $\tilde{\zeta}_j$. By 1 in Lemma 2.6, from (10) we obtain

$$\tilde{\zeta}_j \circ f = \tilde{\zeta}_i \ . \tag{11}$$

By (11) the map f factors an injection — namely, $\tilde{\zeta}_i$ — so it must be injective. ☐

Remark 5.2. In the proof of preceding lemma we used the obvious fact that ζ is a covering map. Now observe further that $\mathfrak{t}_\infty$ is homeomorphic to the real line $\mathbb{R}$ (with its Euclidean topology), and that the homeomorphism $\rho\colon \mathfrak{t}_\infty \to \mathbb{R}$ can be chosen so that each $\mathfrak{t}_i \subseteq \mathfrak{t}_\infty$ is thrown by ρ onto the interval $[-i, i] \subseteq \mathbb{R}$, for each integer $i \geqslant 1$. Since $\mathfrak{B}$ is homeomorphic to the unit circle $\mathbb{S}^1$, we see that, topologically, ζ is a distinguished covering map of $\mathbb{S}^1$ — it is (a piecewise linear model of) the universal covering map of the circle. Indeed, in light of our next result it would not be inappropriate to call ζ the *universal covering $\mathbb{Z}$-map of $\mathfrak{B}$*.

Lemma 5.3 (Lifts of $\mathbb{Z}$-maps)**.** *Let $\zeta\colon \mathfrak{t}_\infty \twoheadrightarrow \mathfrak{B}$ be the covering map of $\mathfrak{B}$ as in the above. Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron, for some integer $n \geqslant 0$, and let $\lambda\colon P \to \mathfrak{B}$ be a $\mathbb{Z}$-map. Then the following hold.*

1. *Any lift $\tilde\lambda\colon P \to \mathfrak{t}_\infty$ of λ is a $\mathbb{Z}$-map.*
2. *If P is connected, and $\tilde\lambda, \tilde\lambda'\colon P \to \mathfrak{t}_\infty$ are two lifts of λ that agree at one point of P, then $\tilde\lambda = \tilde\lambda'$.*
3. *If P is simply-connected, then a lift of λ does exist. In fact, for any point $p \in P$, and for any point $\tilde t \in \mathfrak{t}_\infty$ lying in the fibre over $\lambda(p)$, i.e. such that $\zeta(\tilde t) = \lambda(p)$, there is a lift $\tilde\lambda$ of λ such that $\tilde\lambda(p) = \tilde t$.*

Proof. 1. Let $\tilde\lambda\colon P \to \mathfrak{t}_\infty$ be a lift of λ. Let us display $\tilde\lambda$ and λ in scalar components as

$$\tilde\lambda = (\tilde\lambda_1, \tilde\lambda_2, \tilde\lambda_3)\ ,$$
$$\lambda = (\lambda_1, \lambda_2)\ ,$$

where each $\lambda_i\colon P \to \mathbb{R}$ is a $\mathbb{Z}$-map, $i = 1, 2$, and each $\tilde\lambda_i\colon P \to \mathbb{R}$ is continuous, $i = 1, 2, 3$. By the hypothesis

$$\zeta \circ \tilde\lambda = \lambda$$

together with the definition of ζ we obtain, for each $p \in P$,

$$\zeta(\tilde\lambda(p)) = \zeta(\tilde\lambda_1(p), \tilde\lambda_2(p), \tilde\lambda_3(p)) = (\tilde\lambda_1(p), \tilde\lambda_2(p)) = (\lambda_1(p), \lambda_2(p))\ ,$$

so that $\tilde\lambda_1 = \lambda_1$ and $\tilde\lambda_2 = \lambda_2$. This shows that $\tilde\lambda_1$ and $\tilde\lambda_2$ are $\mathbb{Z}$-maps. All that remains to be shown is that $\tilde\lambda_3$ is a $\mathbb{Z}$-map, too.

Since P is a polyhedron, it is compact. An easy exercise shows that any continuous image of a compact space is compact, so that $\tilde\lambda(P)$ is a compact subspace of $\mathfrak{t}_\infty$. By Remark 5.2, the space $\mathfrak{t}_\infty$ is homeomorphic to the real line with its Euclidean topology via the homeomorphism $\rho\colon \mathfrak{t}_\infty \to \mathbb{R}$; by the Heine-Borel theorem, $\rho(\tilde\lambda(P))$ is closed and bounded in $\mathbb{R}$. It then follows at once from the fact that ρ is a homeomorphism that there are two points $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathfrak{t}_\infty$ with the property that for every $p \in P$ we have $a_3 \leqslant \tilde\lambda_3(p) \leqslant b_3$. In fact, it is evidently possible to choose $a = (1, 0, z_1)$ and $b = (1, 0, z_2)$, for two integers $z_1 < z_2$, in such a way that

$$z_1 < \tilde\lambda_3(p) < z_2 \ \text{ holds for each } p \in P\ . \tag{12}$$

Now set

$$H = [z_1, z_2] \cap \mathbb{Z}\ .$$

Claim. *There exists a finite set of linear $\mathbb{Z}$-maps $l_i\colon P \to \mathbb{R}$, $i \in \{1, \ldots, u\}$, such that for any point $p \in P$ we have*

$$\tilde\lambda_3(p) = z_p + l_{i_p}(p) \tag{13}$$

for some choice of $z_p \in H$ and $i_p \in \{1, \ldots, u\}$.

Proof of Claim. Because λ_1 and λ_2 are $\mathbb{Z}$-maps, there exists a finite set of linear polynomials with integer coefficients $l_i\colon \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \ldots, u\}$, such that, at each $p \in P$, λ_1 agrees with some l_{i_1}, and λ_2 with some l_{i_2}. We have $\lambda(p) = (\lambda_1(p), \lambda_2(p)) \in \mathfrak{B}$.

Case 1. $\lambda_2(p) = 0$. Then the fibre over $\lambda(p)$ consists of the set

$$\zeta^{-1}(\lambda(p)) = \{\big(\,\lambda_1(p),\, 0,\, z + \lambda_1(p)\,\big) \in \mathbb{R}^3 \mid z \in \mathbb{Z}\}\,.$$

But λ_1 agrees with l_{i_p} at p, for some $i_p \in \{1, \ldots, u\}$; therefore,

$$\zeta^{-1}(\lambda(p)) = \{\big(\,l_{i_p}(p),\, 0,\, z + l_{i_p}(p)\,\big) \in \mathbb{R}^3 \mid z \in \mathbb{Z}\}\,.$$

Since $\tilde{\lambda}$ is a lift of λ, $\tilde{\lambda}(p) = (\lambda_1(p), \lambda_2(p), \tilde{\lambda}_3(p))$ must lie in the fibre over $\lambda(p)$. Thus,

$$\tilde{\lambda}_3(p) = z_p + l_{i_p}(p) \ \text{ for some } z_p \in \mathbb{Z}\ . \tag{14}$$

By (12) and (14) we obtain

$$z_1 < z_p + l_{i_p}(p) < z_2\ . \tag{15}$$

Since $(l_{i_p}(p), \lambda_2(p)) \in \mathfrak{B}$, we have $0 \leqslant l_{i_p}(p) \leqslant 1$, so that (15) implies $z_p \in H$, as was to be shown.

Case 2. $\lambda_2(p) \neq 0$. Then the fibre over $\lambda(p)$ consists of the set

$$\zeta^{-1}(\lambda(p)) = \{(\,\lambda_1(p),\, \lambda_2(p),\, z\,) \in \mathbb{R}^3 \mid z \in \mathbb{Z}\}\,.$$

Since $\tilde{\lambda}$ is a lift of λ, $\tilde{\lambda}(p) = (\lambda_1(p), \lambda_2(p), \tilde{\lambda}_3(p))$ must lie in the fibre over $\lambda(p)$. Thus,

$$\tilde{\lambda}_3(p) = z_p \ \text{ for some } z_p \in \mathbb{Z}\ . \tag{16}$$

By (12) and (16) we obtain

$$z_1 < z_p < z_2\ ,$$

so that $z_p \in H$, as was to be shown. □

Because the expression on the right-hand side of (13) denotes a linear $\mathbb{Z}$-map, the Claim entails at once that there is a finite collection of linear polynomials $\mathbb{R}^n \to \mathbb{R}$ with integer coefficients — namely, those of the form $z + l$, for $z \in H$ and $l \in \{l_1, \ldots, l_u\}$ — such that for each point $p \in P$, $\tilde{\lambda}_3(p)$ agrees with one of them. By Lemma 2.4 this is equivalent to saying that $\tilde{\lambda}_3$ is a $\mathbb{Z}$-map, as was to be shown.

2. By 1 in Lemma 2.6.
3. By 2 in Lemma 2.6. □

6. Proof of Theorem.

Lemma 6.1. *For each integer $i \geqslant 1$, there is an integer $n_i \geqslant 1$ such that $\mathfrak{t}_i$ is $\mathbb{Z}$-homeomorphic to a retract of $[0,1]^{n_i}$ by $\mathbb{Z}$-maps.*

Proof. Let d be the number of lattice points in $\mathfrak{t}_i$, and let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. Consider the (fundamental) simplex $|\Delta| = \texttt{conv}\,\{e_1, \ldots, e_d\} \subseteq [0,1]^d$ in $\mathbb{R}^d$, and let Δ be the unimodular simplicial complex whose collection of simplices consists of all faces of $|\Delta|$. Write $e_{h,k}$ for the 1-simplex in $\mathbb{R}^d$ given by $\texttt{conv}\,\{e_h, e_k\}$, for $h \neq k \in \{1, \ldots, d\}$. Set

$$\Pi = \{\emptyset\}\ \cup\ \{e_1, \ldots, e_d\}\ \cup \{e_{1,2},\, e_{2,3},\, \ldots,\, e_{d-1,d}\}\ .$$

Then Π is a unimodular simplicial complex in $[0,1]^d$. We claim that $|\Pi|$ is $\mathbb{Z}$-homeomorphic to $\mathfrak{t}_i$. To see this, let

$$z_1, \ldots, z_d$$

be the lattice points in t_i, listed in the order they are encountered when traversing t_i from $(1, 0, -i)$ to $(1, 0, i)$. Let $f\colon |\Pi| \to \mathsf{t}_i$ be the unique continuous map that extends the correspondence

$$e_j \mapsto z_j \ , \ j = 1, \ldots, d \ ,$$

and is linear over each simplex of $|\Pi|$. By construction, f is a homeomorphism. A trivial computation in linear algebra then shows that both f and f^{-1} are $\mathbb{Z}$-maps, so that $|\Pi|$ and t_i indeed are $\mathbb{Z}$-homeomorphic.

It remains to show that $[0, 1]^d$ retracts onto $|\Pi|$. This follows from an application of Lemma 3.8, upon observing that $|\Pi|$ evidently is a strongly regular polyhedron of dimension 1. $\square$

Lemma 6.2. (i) *The following diagram commutes for every integer $i \geqslant 1$.*

$$\begin{tikzcd}
& \mathsf{t}_i \arrow[dl, two heads, "\zeta_i"'] \arrow[dd, hook, "\eta_i"] \\
\mathfrak{B} & \\
& \mathsf{t}_{i+1} \arrow[ul, two heads, "\zeta_{i+1}"]
\end{tikzcd}$$

(ii) *For any two integers $i > j > 0$, there is no $\mathbb{Z}$-map $\lambda\colon \mathsf{t}_i \to \mathsf{t}_j$ making the following diagram commute.*

$$\begin{tikzcd}
& \mathsf{t}_i \arrow[dl, two heads, "\zeta_i"'] \arrow[dd, "\lambda"] \\
\mathfrak{B} & \\
& \mathsf{t}_{j} \arrow[ul, two heads, "\zeta_{j}"]
\end{tikzcd}$$

(iii) *For an integer $n \geqslant 1$, suppose $P \subseteq [0, 1]^n$ is a rational polyhedron that is a retract of $[0, 1]^n$ by $\mathbb{Z}$-maps, and let $\lambda\colon P \to \mathfrak{B}$ be any $\mathbb{Z}$-map. Then there exist an integer $i_0 \geqslant 1$ and a $\mathbb{Z}$-map $\lambda'\colon P \to \mathsf{t}_{i_0}$ such that the following diagram commutes.*

$$\begin{tikzcd}
& \mathsf{t}_{i_0} \arrow[dl, two heads, "\zeta_{i_0}"'] \\
\mathfrak{B} & \\
& P \arrow[ul, "\lambda"] \arrow[uu, "\lambda'"']
\end{tikzcd}$$

Proof. (i) By direct inspection of the definitions.

(ii) Since $i > j > 0$, t_i contains strictly more lattice points than t_j by construction. Since $\mathbb{Z}$-maps carry lattice points to lattice points, any λ as in the statement would fail to be injective. By Lemma 5.1, no such map can exist.

(iii) Let us first show that P is simply-connected. It is clear that $[0, 1]^n$ is connected, and connectedness is obviously preserved by continuous maps; so P is connected. Now, $\pi_1([0, 1]^n)$ is the trivial group $\{*\}$, because $[0, 1]^n$ is convex [16, Example 1.4]. Further, let $r\colon [0, 1]^n \twoheadrightarrow P$ be a retraction of $[0, 1]^n$ onto P, and let $s\colon P \hookrightarrow [0, 1]^n$ be a section of r, so that $r \circ s = \mathbf{1}_P$. Then by functoriality the pair r, s induces the following commutative diagram

$$\begin{tikzcd}
\pi_1(P) \arrow[r, "\pi_1(s)"] \arrow[rr, bend right, "\mathbf{1}_{\pi_1(P)}"'] & \{*\} \arrow[r, "\pi_1(r)"] & \pi_1(P)
\end{tikzcd}$$

so that $\pi_1(P)$ is the trivial group, too.

Now 3 in Lemma 5.3 yields the existence of a $\mathbb{Z}$-map $\tilde{\lambda}\colon P \to \mathfrak{t}_\infty$ satisfying $\zeta \circ \tilde{\lambda} = \lambda$. Since P is compact, and since a continuous image of a compact set is compact, from the fact that $\mathfrak{t}_\infty = \bigcup_{i \geqslant 1} \mathfrak{t}_i$ it immediately follows that there must exist an integer $i_0 \geqslant 1$ such that $\tilde{\lambda}(P) \subseteq \mathfrak{t}_{i_0}$. Let $\lambda'\colon P \to \mathfrak{t}_{i_0}$ be the co-restriction of $\tilde{\lambda}$ to $\mathfrak{t}_{i_0}$. Then λ' is a $\mathbb{Z}$-map that satisfies $\zeta_{i_0} \circ \lambda' = \lambda$, as was to be shown. □

End of proof of Theorem. Let $\mathcal{F}_2$ be the MV-algebra freely generated by the elements X_1, X_2. Let θ be the congruence relation on $\mathcal{F}_2$ generated by the pair $(X_1 \vee \neg X_1 \vee X_2 \vee \neg X_2, \top)$. Since θ is principal by construction, $A = \mathcal{F}_2/\theta$ is finitely presented. The solution set of θ in $[0,1]^2$ (in the sense of $(1 \Leftrightarrow 4)$ in Lemma 3.2) is $\mathfrak{B}$. By Lemmas 3.1 and 3.2 together with the duality theorem, therefore, we see that $\mathscr{M}(\mathfrak{B}) \cong A$, and that the inclusion map $\mathfrak{B} \hookrightarrow [0,1]^2$ is dual to the quotient map $\mathcal{F}_2 \twoheadrightarrow A$. Set

$$
\begin{aligned}
T_i &= \mathscr{M}(\mathfrak{t}_i)\,,\\
u_i &= \mathscr{M}(\zeta_i)\colon A \hookrightarrow T_i\,,
\end{aligned}
$$

for each $i \geqslant 1$. By Lemma 6.1 and Corollary 3.6, each T_i is finitely presented and projective. By the duality theorem and Lemma 6.2, $\{u_i\}$ is a strictly increasing chain of order-type ω of algebraic unifiers for A that is co-final in the partially ordered set of algebraic unifiers for A. The proof is completed by an application of Theorem 2.1. □

7. Conclusions and further research.

We have proved that the unification type of MV-algebras is nullary, and that nullary unification problems already occur over two variables. The unification type of Łukasiewicz logic was first shown to be non-unitary by Dzik [11, Corollary 11]. Indeed, our unification problem of nullary type $(\star)$ may be regarded as a bivariate generalisation of the one-variable problem used by Dzik to prove non-unitarity — the *tertium non datur* principle $X \vee \neg X = 1$.

Unification theory can be applied to the study of admissible rules, see e.g. [15]. Jeřábek [17, 18] provides an explicit basis for admissible rules in Łukasiewicz logic, proves that the set of admissible rules is decidable, and shows that no finite basis exists.

Subvarieties of MV-algebras (=schematic extensions of Łukasiewicz logic) have been completely classified by Komori [19]; see also [10, 8.4]. Dzik's main result [11, Theorem 9] entails as a special case that each subvariety of MV-algebras generated by a single finite chain (see [10, 8.5.2]) has unitary unification type. Can one use Komori's classification to determine the unification type of each subvariety of MV-algebras?

Since MV-algebras are categorically equivalent to lattice-ordered Abelian groups with a strong order unit [23], the Theorem translates easily to the latter context upon using the category-theoretic notion of unification type outlined in Remark 2.2 above. (An easy compactness argument shows that lattice-ordered Abelian groups with a strong order unit are not an elementary class, so that recourse to Remark 2.2 is unavoidable.) The situation for lattice-ordered Abelian groups (without a distinguished unit) is different. Beynon proved [4, Theorem 3.1] that the finitely generated projective lattice-ordered Abelian groups are exactly the finitely presented ones. Hence the unification type of the theory of lattice-ordered Abelian groups is unitary.

One would like to have a deeper understanding of finitely presented projective MV-algebras and their dual rational polyhedra; what is known at present — essentially, the results of [7] — shows that the difficulties involved are not to be taken lightly.

Trivially, it is uniformly decidable whether a unification problem in the language of MV-algebras is unifiable: one shows first that it suffices to check if some ground substitution that replaces variables by the constants 0 and 1 unifies the problem; and then that the latter condition is decidable by classical truth-tables, because the only MV-algebraic structure on $\{0,1\}$ is Boolean. By contrast, almost any other significant decision problem for MV-algebras seems to be open — with the exception of the word problem, for which see [25]. For example, the isomorphism problem for finitely presented MV-algebras is open. Also, it is open whether projectivity of a finite presentation can be algorithmically recognised. Similarly, it is open whether there is an algorithm to compute the unification type of a unification problem, or even to separate problems with nullary type from the remaining ones. Much remains to be understood.

In the proof of the Theorem we use the fact that the number of lattice points in the polyhedra $\mathfrak{t}_i$ is an unbounded function of i, whereas any retract of $[0,1]^n$ by $\mathbb{Z}$-maps has at most 2^n lattice points. This implies that the strictly increasing chain of unifiers that witnesses the nullary type of ($\star$) takes advantage of a countable infinity of variables. It is natural to ask whether the unification type improves for fragments of Łukasiewicz logic restricted to a finite number of variables. Such fragments have a corresponding E-unification theory, of course. Continuing with the notation adopted in the Introduction, one simply takes $\mathscr{V}$ to be a finite set, and substitutions to be arbitrary maps $\sigma\colon \mathscr{V} \to \mathsf{Term}_{\mathscr{V}}(\mathscr{F})$. The terms occurring in a unification problem $\mathscr{E}$ are constrained to come from $\mathsf{Term}_{\mathscr{V}}(\mathscr{F})$, too. Unifiers and the unification type are defined in the obvious fashion. In light of the fact that the duality theorem of Section 3 specialises to a duality between MV-algebras finitely presented over n generators, and rational polyhedra contained in $[0,1]^n$, tools similar to the ones used in this paper can be applied to the investigation of n-variable fragments of Łukasiewicz logic. We close with a conjecture.

Conjecture. *The unification type of Łukasiewicz logic restricted to n variables is nullary, for each integer $n \geqslant 2$.*

Acknowledgements. The first-named author is indebted to Silvio Ghilardi for a clarifying discussion related to footnote 2 of the present paper. The second-named author wishes to thank Antonio Di Nola and Revaz Grigolia for introducing him to the topic of this article and for their useful comments on unification problems in Łukasiewicz logic. Both authors are especially indebted to Emil Jeřábek for having pointed out to them a serious blunder in a preliminary version of this paper. Finally, both authors were partially supported by the Italian National Research Project (PRIN2008) entitled *Logica ed algebra dell'informazione incerta.*

References

[1] J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1994.

[2] F. Baader and S. Ghilardi. Unification in Modal and Description Logics. To appear.

[3] F. Baader and W. Snyder. Unification theory. In J. A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, pages 445–532. Elsevier and MIT Press, 2001.

[4] W. M. Beynon. Applications of duality in the theory of finitely generated lattice-ordered abelian groups. *Canad. J. Math.*, 29(2):243–254, 1977.

[5] S. Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1981.

[6] L. M. Cabrer and D. Mundici. Projective MV-algebras and rational polyhedra. *Algebra universalis*, 62(1):63–74, 2009.

[7] L. M. Cabrer and D. Mundici. Rational polyhedra and projective lattice-ordered abelian groups with order unit. *ArXiv preprint arXiv:0907.3064*, 2009. To appear in *Commun. Contemp. Math.*

[8] C. C. Chang. Algebraic analysis of many valued logic. *Trans. Amer. Math. Soc.*, 88:467–490, 1958.

[9] C. C. Chang. A new proof of the completeness of the łukasiewicz axioms. *Trans. Amer. Math. Soc.*, 93:74–80, 1959.

[10] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic foundations of many-valued reasoning*, volume 7 of *Trends in Logic—Studia Logica Library.* Kluwer Academic Publishers, Dordrecht, 2000.

[11] W. Dzik. Unification in some substructural logics of BL-algebras and hoops. *Reports in Mathematical Logic*, 43:73–83, 2008.

[12] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien.* Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin, 1971.

[13] W. J. Blok and D. Pigozzi. *Algebraizable logics.* Mem. Amer. Math. Soc., Vol. 77. The American Mathematical Society, Providence, 1989.

[14] S. Ghilardi. Unification through projectivity. *J. Logic Comput.*, 7(6):733–752, 1997.

[15] S. Ghilardi. Unification in intuitionistic logic. *Journal of Symbolic Logic*, 64(2):859–880, 1999.

[16] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[17] E. Jeřábek. Admissible rules of Łukasiewicz logic. *Journal of Logic and Computation*, 20(2):425–447, 2010.

[18] E. Jeřábek. Bases of admissible rules of Łukasiewicz logic. *Journal of Logic and Computation*, 2010. To appear.

[19] Y. Komori. Super-Łukasiewicz propositional logics. *Nagoya Math. J.*, 84:119–133, 1981.

[20] J. Łukasiewicz and A. Tarski. Untersuchngen über den Aussagenkalkül. *C. R. Soc. Sc. Varsovie*, 23:30–50, 1930.
[21] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
[22] V. Marra and D. Mundici. The Lebesgue state of a unital abelian lattice-ordered group. *J. Group Theory*, 10(5):655–684, 2007.
[23] D. Mundici. Interpretation of AF C^* algebras in Łukasiewicz sentential calculus. *Journal of Functional Analysis*, 65:15–63, 1986.
[24] D. Mundici. Farey stellar subdivisions, ultrasimplicial groups, and K_0 of AF C^*-algebras. *Adv. in Math.*, 68(1):23–39, 1988.
[25] D. Mundici. *Advanced Łukasiewicz Calculus and MV-algebras*, volume 35 of *Trends in Logic—Studia Logica Library*. Springer, New York, 2011.
[26] G. Plotkin. Building-in equational theories. *Machine intelligence*, 7(73-90):30, 1972.
[27] J. A. Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12(1):23–41, 1965.
[28] A. Rose and J. B. Rosser. Fragments of many-valued statement calculi. *Trans. Amer. Math. Soc.*, 87:1–53, 1958.
[29] C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology*. Springer Study Edition. Springer-Verlag, Berlin, 1982. Reprint.
[30] C. L. Siegel. *Lectures on the geometry of numbers*. Springer-Verlag, Berlin, 1989. Notes by B. Friedman, rewritten by K. Chandrasekharan with the assistance of R. Suter.
[31] A. Tarski. *Logic, semantics, metamathematics. Papers from 1923 to 1938*. Oxford at the Clarendon Press, 1956. Translated by J. H. Woodger.
[32] J. van Heijenoort, editor. *From Frege to Gödel*. Harvard University Press, Cambridge, MA, 2002. A source book in mathematical logic, 1879–1931. Reprint of the third printing of the 1967 original.