

Homogeneous orthocomplete effect algebras are covered by MV-algebras

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- 1 Introduction
- 2 Basic definitions
- 3 The condition (W_+)
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Joint work of J. Niederle and J. Paseka

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The aim of our paper is to show that every block of an Archimedean homogeneous effect algebra satisfying the property (W_+) is lattice ordered. Therefore, any Archimedean homogeneous effect algebra satisfying the property (W_+) is covered by MV-algebras.

As a corollary, this yields that every block of a homogeneous orthocomplete effect algebra is a Heyting effect algebra.

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Basic definition – effect algebras

Definition (D. Foulis and M.K. Bennett, 1994)

A partial algebra $(E; \oplus, 0, 1)$ is called an **effect algebra** if $0, 1$ are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

- (Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,
- (Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,
- (Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),
- (Eiv) if $1 \oplus x$ is defined then $x = 0$.

Example

Let $E = [0, 1] \subseteq \mathbb{R}$. We put $x \oplus y = x + y$ iff $x + y \leq 1$. Hence $\frac{3}{4} \oplus \frac{4}{5}$ does not exist in E .

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A subset $Q \subseteq E$ is called a *sub-effect algebra* of E if

- (i) $1 \in Q$
- (ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q , then $x, y, z \in Q$.

An effect algebra E is called an *orthoalgebra* if $x \oplus x$ exists implies that $x = 0$.

On every effect algebra E the partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x \leq y \text{ and } y \ominus x = z \text{ iff } x \oplus z \text{ is defined and } x \oplus z = y.$$

If E with the defined partial order is a (complete) lattice then $(E; \oplus, 0, 1)$ is called a (*complete*) *lattice effect algebra*.

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An effect algebra E satisfies the *Riesz decomposition property* (or RDP) if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2$, there are u_1, u_2 such that $u_1 \leq v_1, u_2 \leq v_2$ and $u = u_1 \oplus u_2$.

- (i) Every lattice effect algebra with RDP can be organized into an MV-algebra and conversely.
- (ii) Every MV-algebra which is an orthoalgebra is a Boolean algebra.

An effect algebra E is called *homogeneous* if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2 \leq u'$, there are u_1, u_2 such that $u_1 \leq v_1, u_2 \leq v_2$ and $u = u_1 \oplus u_2$.

A subset B of E is called a *block* of E if B is a maximal sub-effect algebra of E with the Riesz decomposition property.

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Basic definitions – Effect algebras

- (i) Every orthoalgebra is homogeneous.
- (ii) Every lattice effect algebra is homogeneous.
- (ii) Every effect algebra with the Riesz decomposition property is homogeneous.

An element w of an effect algebra E is called *sharp* if $w \wedge w' = 0$.

The well known fact is that in every lattice effect algebra E the subset $S(E) = \{w \in E \mid w \wedge w' = 0\}$ is a sub-lattice effect algebra of E being an orthomodular lattice.

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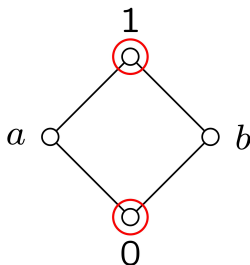
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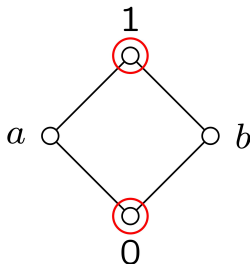
Blocks of E and $S(E)$ - Example 1 - The diamond



$$a \oplus a = b \oplus b = 1$$

- $S(E) = \{0, 1\}$ is a Boolean algebra, but E has two blocks, $\{0, a, 1\}$ and $\{0, b, 1\}$.
- For any block B of E , $S(E) \cap B = \{0, 1\}$ is a block of $S(E)$.

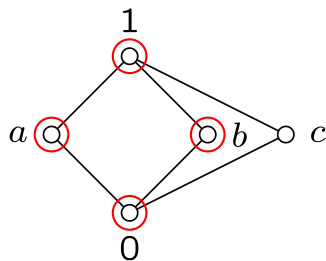
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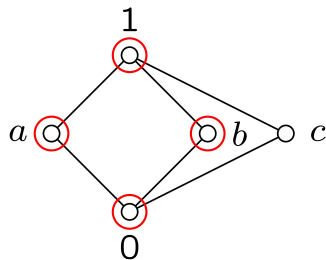
Blocks of E and $S(E)$ - Example 2



$$a \oplus b = c \oplus c = 1$$

- $S(E) = \{0, a, b, 1\}$ is a Boolean algebra and E has again two blocks. Namely, there are two blocks here, the Boolean algebra $S(E)$ and a 3-element chain $C_3 = \{0, c, 1\}$.
- $S(E) \cap C_3 = \{0, 1\}$ is not a block of $S(E)$.

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Meager and hypermeager elements

In what follows set

$$M(E) = \{x \in E \mid \text{if } v \in S(E) \text{ satisfies } v \leq x \text{ then } v = 0\}.$$

We also define

$$HM(E) = \{x \in E \mid \text{there is } y \in E \text{ such that } x \leq y \text{ and } x \leq y'\}$$

and

$$UM(E) = \{x \in E \mid \text{for every } y \in S(E) \text{ such that } x \leq y \text{ it holds } x \leq y \oplus x\}.$$

An element $x \in M(E)$ is called *meager*, an element $x \in HM(E)$ is called *hypermeager* and an element $x \in UM(E)$ is called *ultrameager*.

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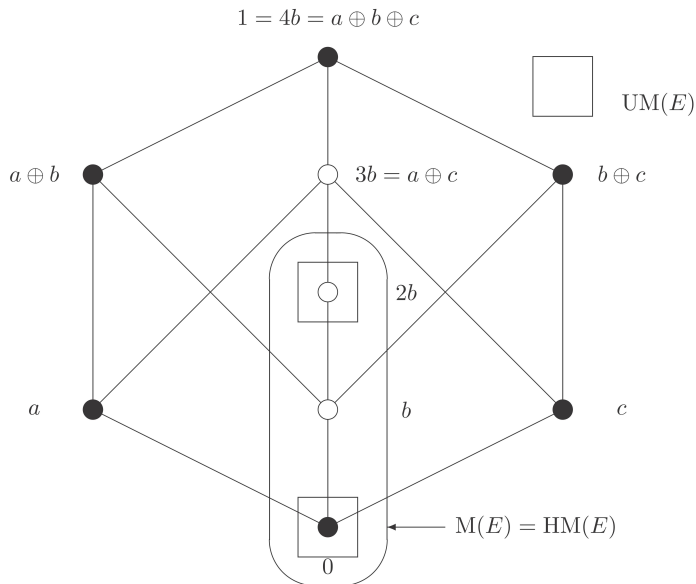
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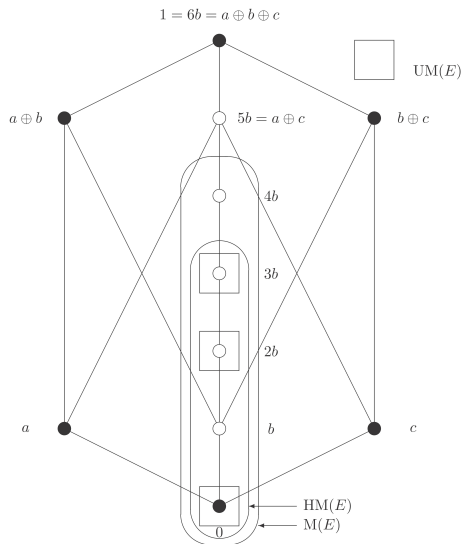
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Meager and hypermeager elements - Example 3



Meager and hypermeager elements - Example 4



Meager and hypermeager elements

Lemma

Let E be an effect algebra. Then $\text{UM}(E) \subseteq \text{HM}(E) \subseteq \text{M}(E)$. Moreover, for all $x \in E$, $x \in \text{HM}(E)$ iff $x \oplus x$ exists and, for all $y \in \text{M}(E)$, $y \neq 0$ there is $h \in \text{HM}(E)$, $h \neq 0$ such that $h \leq y$.

Lemma

In every homogeneous effect algebra E , $\text{UM}(E) = \text{HM}(E)$.

For an element x of an effect algebra E we write $\text{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n -times) exists for every positive integer n and we write $\text{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E . An effect algebra E is *Archimedean* if $\text{ord}(x) < \infty$ for all $x \in E$, $x \neq 0$.

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Orthogonal systems

We say that a finite system $F = (x_k)_{k=1}^n$ of not necessarily different elements of an effect algebra E is *orthogonal* if $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ (written $\bigoplus_{k=1}^n x_k$ or $\bigoplus F$) exists in E .

An arbitrary system $G = (x_\kappa)_{\kappa \in H}$ of not necessarily different elements of E is called *orthogonal* if $\bigoplus K$ exists for every finite $K \subseteq G$.

We say that for a orthogonal system $G = (x_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee \{\bigoplus K \mid K \subseteq G \text{ is finite}\}$ exists in E and then we put $\bigoplus G = \bigvee \{\bigoplus K \mid K \subseteq G \text{ is finite}\}$. We say that $\bigoplus G$ is the *orthogonal sum* of G and G is *orthosummable*. (Here we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (x_\kappa)_{\kappa \in H_1}$). We denote $G^\oplus := \{\bigoplus K \mid K \subseteq G \text{ is finite}\}$.

E is called *orthocomplete* if every orthogonal system is orthosummable.

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- 2 Basic definitions
- 3 The condition (W_+)**
- 4 Main theorem and its generalizations

The condition (W_+)

An effect algebra E fulfills the condition (W_+) (introduced by Tkadlec) if for each orthogonal subset $A \subseteq E$ and each two upper bounds u, v of $A^\oplus$ there exists an upper bound w of $A^\oplus$ below u, v .

Statement (Tkadlec 2010)

Lattice effect algebras and orthocomplete effect algebras fulfill the condition (W_+).

Every orthocomplete effect algebra is Archimedean.

Proposition

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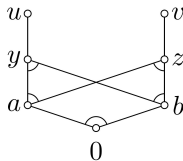
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Shifting lemma

Lemma (Shifting lemma)

Let E be an Archimedean effect algebra fulfilling the condition (W_+) , let $u, v \in E$, and let a_1, b_1 be two maximal lower bounds of u, v . There exist elements y, z and two maximal lower bounds a, b of y, z for which $y \leq u$, $z \leq v$, $a \leq a_1$, $b \leq b_1$, $a \wedge b = 0$, a, b are maximal lower bounds of y, z and y, z are minimal upper bounds of a, b . Furthermore, $(y \ominus a) \wedge (z \ominus a) = 0$, $(y \ominus b) \wedge (z \ominus b) = 0$, $(y \ominus a) \wedge (y \ominus b) = 0$, $(z \ominus a) \wedge (z \ominus b) = 0$.

The Shifting lemma provides the following *minimax structure*.



Meets in Archimedean homogeneous effect algebra fulfilling the condition (W_+)

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Let E be an Archimedean homogeneous effect algebra fulfilling the condition (W_+) . For every orthogonal elements u, v , $u \wedge v$ and $u \vee_{[0, u \oplus v]} v$ exist and $[0, u \wedge v] \subseteq B$ for every block B containing u or v .

Corollary

Let E be an Archimedean homogeneous effect algebra fulfilling the condition (W_+) . For every element u , $u \wedge u'$ and $u \vee u'$ exist and $[0, u \wedge u'] \subseteq B$ for every block B containing u .

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Main theorem

Theorem

Let E be an Archimedean homogeneous effect algebra fulfilling the condition (W_+) . Then every block in E is a lattice and E can be covered by MV-algebras.

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Let E be an orthocomplete homogeneous effect algebra. Then E can be covered by Heyting MV-effect algebras.

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Generalization of the Main theorem

Definition

An effect algebra E has the *maximality property* if $\{u, v\}$ has a maximal lower bound w for every $u, v \in E$.

It is easy to see that an effect algebra E has the maximality property if and only if $\{u, v\}$ has a maximal lower bound w , $w \geq t$ for every $u, v, t \in E$ such that t is a lower bound of $\{u, v\}$. As noted by Tkadlec E has the maximality property if and only if $\{u, v\}$ has a minimal upper bound w for every $u, v \in E$.

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Theorem

Let E be a homogeneous effect algebra having the maximality property. Then every block B in E is a lattice and E can be covered by MV-algebras.

Corollary (Riečanová 2000)

Let E be a lattice effect algebra. Then E can be covered by MV-algebras which are blocks of E .

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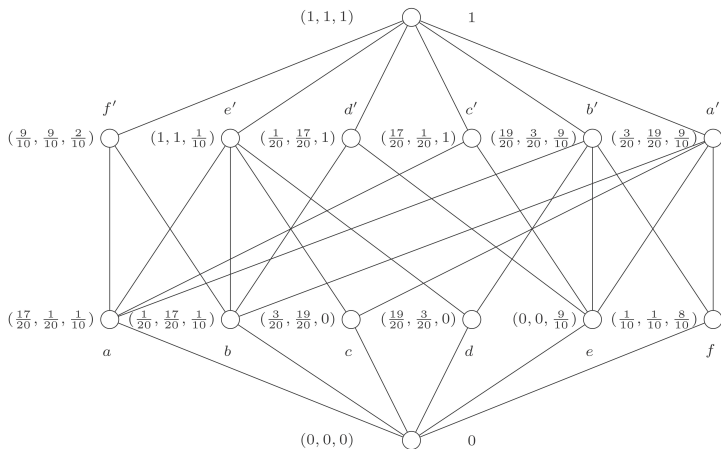
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Example 5 - orthoalgebra E



a and b have two different minimal upper bounds, $a \oplus b = f'$ and $a \oplus c = e'$.

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