

The Modal Logic of the Bi-topological Rational Plane

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(X, τ) a topological space

$$x \in dA \text{ iff } \forall U \in \tau (x \in U \text{ implies } \exists y \in U - \{x\} : y \in A) .$$

d - semantics

$$x \Vdash \Diamond \phi \text{ iff } x \in d(|\phi|)$$

The Main Result

***Theorem.** The modal logic $KD4+KD4$ is sound and complete w.r.t. the bi-topological rational plane $Q \times Q$ with the horizontal and vertical topologies.*

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Theorem. *The modal logic $KD4+KD4$ is sound and complete w.r.t. the bi-topological rational plane $Q \times Q$ with the horizontal and vertical topologies.*

- *In C- Semantics*

Fact (Van Benthem, Bezhanishvili, ten Cate, Sarenac). *The modal logic $S4+S4$ is sound and complete w.r.t. the bi-topological rational plane $Q \times Q$ with the horizontal and vertical topologies.*

J. van Benthem, G. Bezhanishvili, B. ten Cate, and D. Sarenac, Multimodal logics of products of topologies, Studia Logica 84 (2006), no. 3, 369–392.

***Fact (Shehtman).** The modal logic $KD4$ is sound and complete w.r.t. the rational line Q with the standard(interval) topologies*

V. Shehtman, Derived sets in Euclidean spaces and modal logic, Tech. Re- port X-1990-05, Univ. of Amsterdam, 1990.

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Joel Lucero-Bryan – The d-Logic of the Rational
Numbers: A New Proof

KD4+KD4

- The axioms are all classical tautologies, each box satisfies all **KD4** axioms i.e., for each $i \in \{1, 2\}$ we have:

$$(K) \quad \Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q),$$

$$(D) \quad \Box_i p \rightarrow \Diamond_i p,$$

$$(4) \quad \Box_i p \rightarrow \Box_i \Box_i p,$$

- The rules of inference are: Modus-Ponens, Substitution, Necessitation.

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KRIPKE SEMANTICS

Definition 1 *The relation $R \subseteq W \times W$ is:*
serial if - $(\forall x)(\exists y)(xRy)$,
transitive if - $(\forall x, y, z)(xRy \wedge yRz \Rightarrow xRz)$.

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KRIPKE SEMANTICS

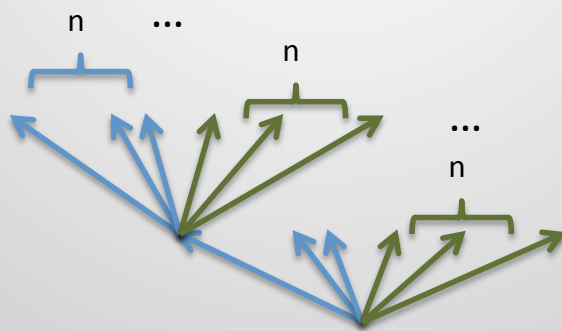
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Proposition The modal logic KD4+KD4 is sound and complete w.r.t. the class of all finite, serial and transitive birelational Kripke structures.

$\not\models \phi$ Then there exists finite model $(W, R_1, R_2, V), w \not\models \phi$

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$(W, R_1, R_2, V), w \not\models \phi$ iff $T_{n,n} \not\models \phi$

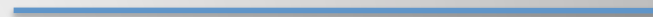


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$(W, R_1, R_2, V), w \not\models \phi$ iff $T_{n,n} \not\models \phi$

$T_{n,n}, w \not\models \phi$ implies $T_{\omega,\omega} \not\models \phi$

Q



Q

0

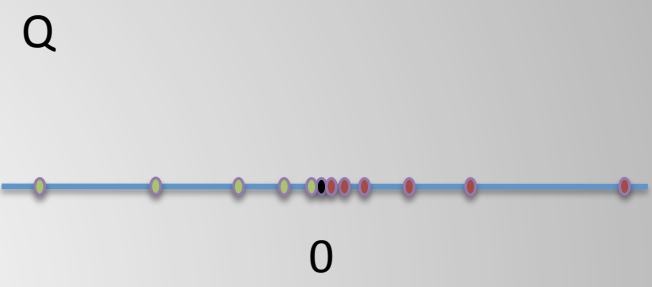


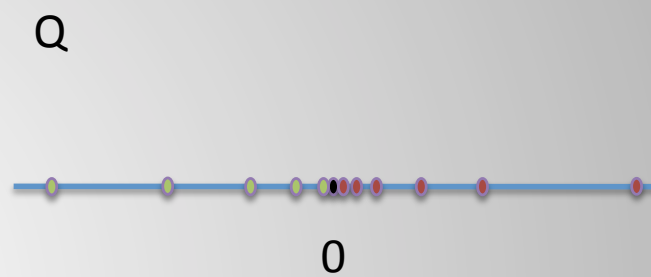
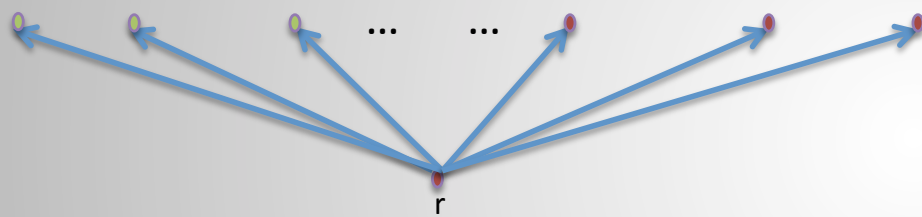
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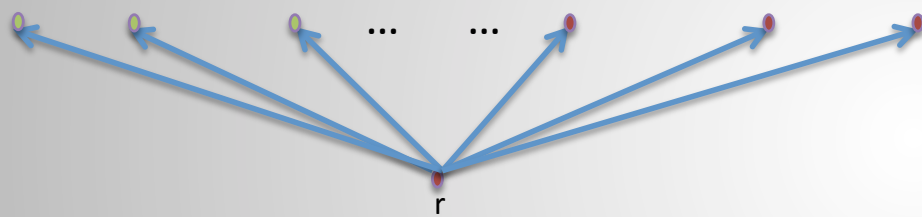
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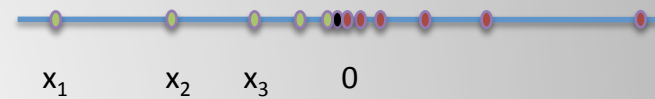
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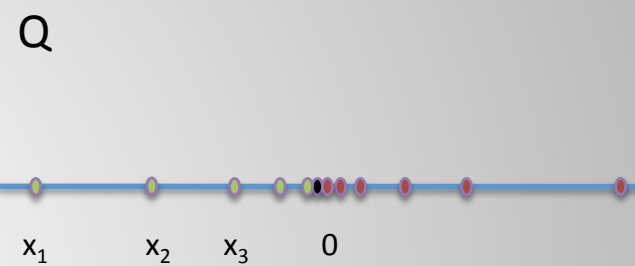
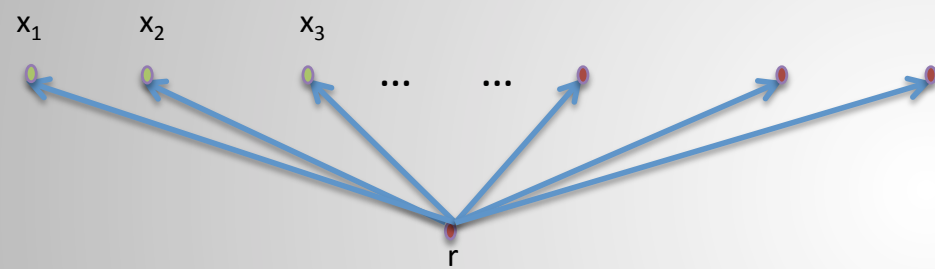


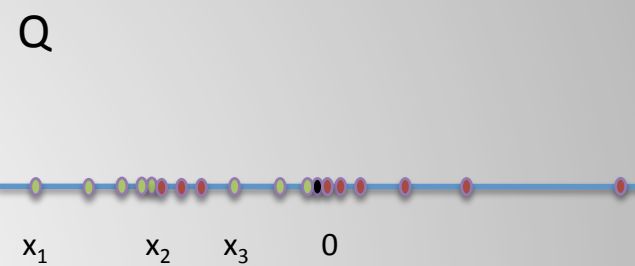
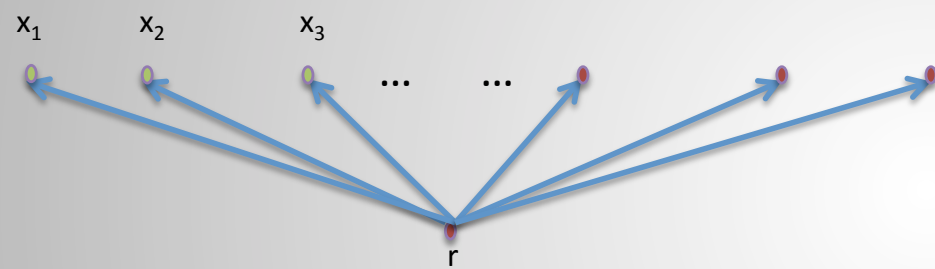


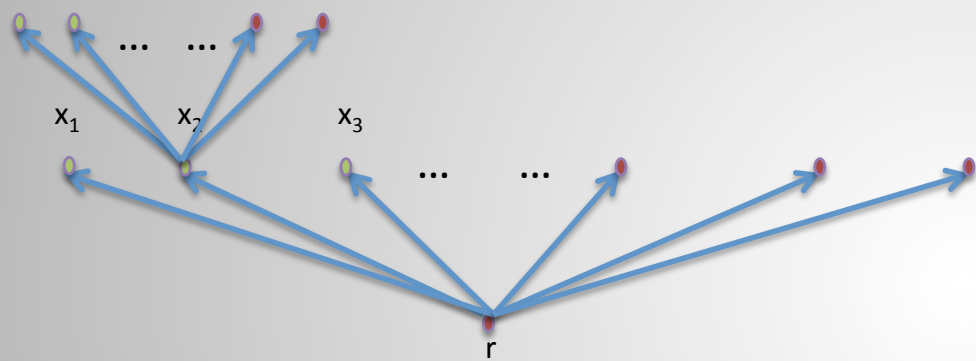


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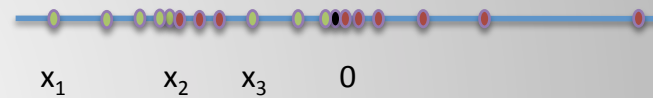








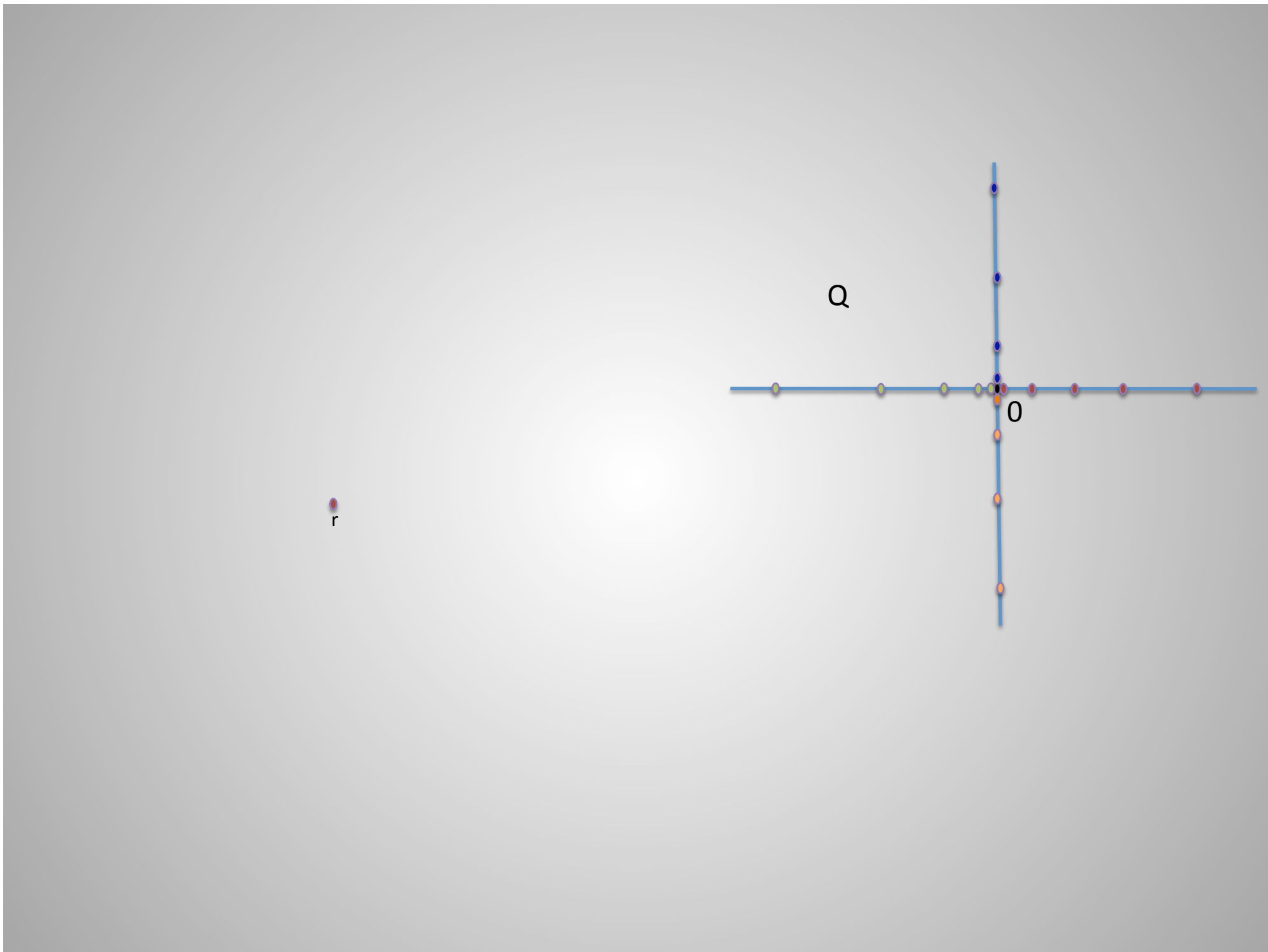
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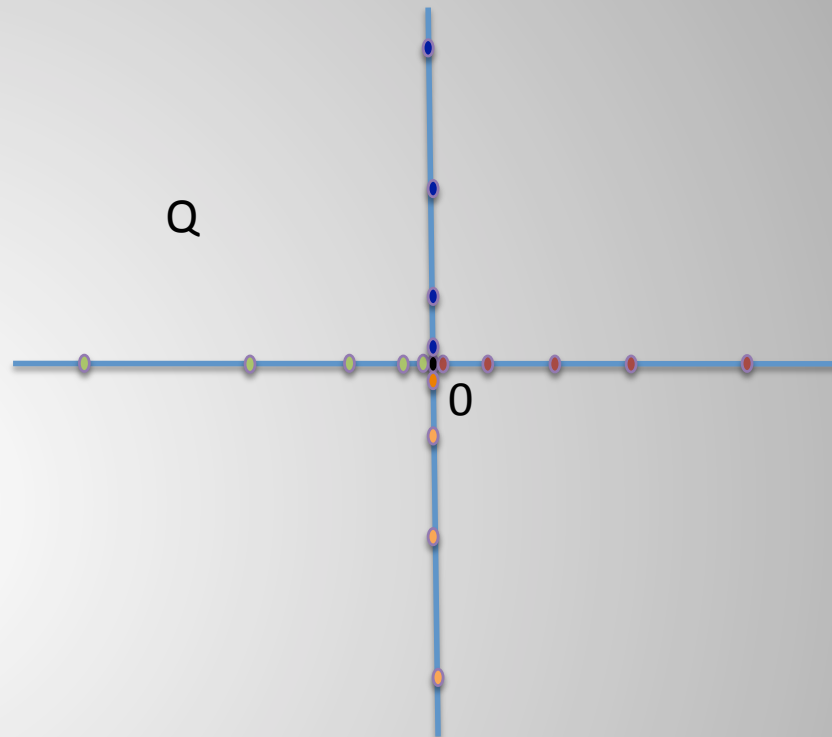
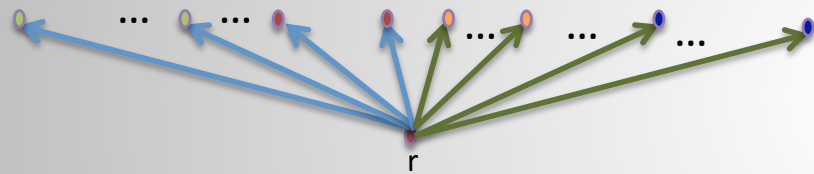


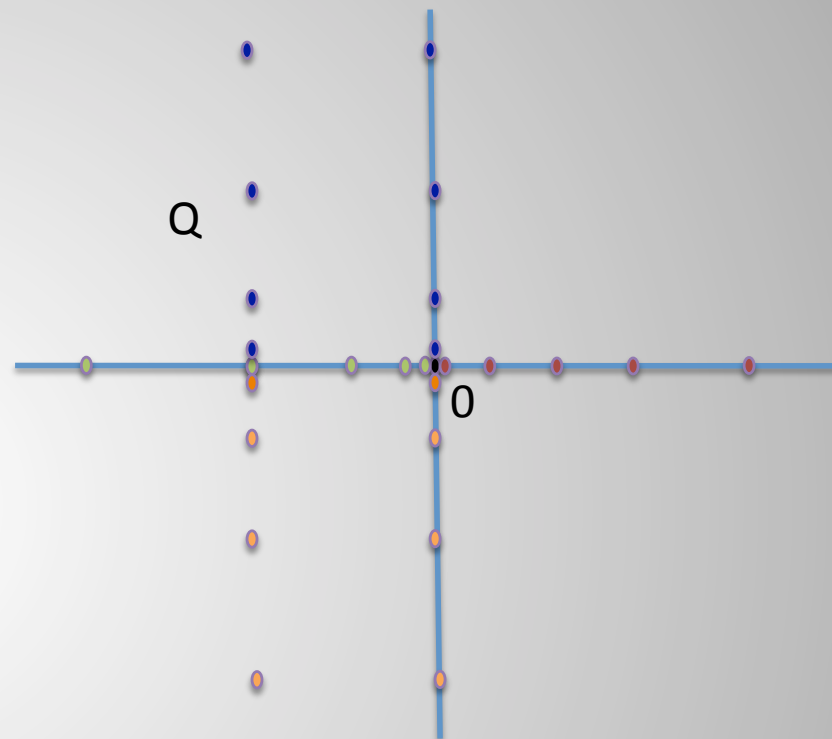
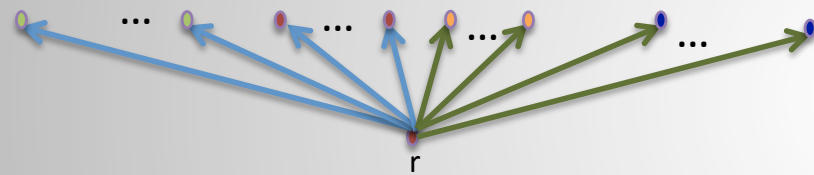
- *The horizontal and vertical topologies in $Q \times Q$*

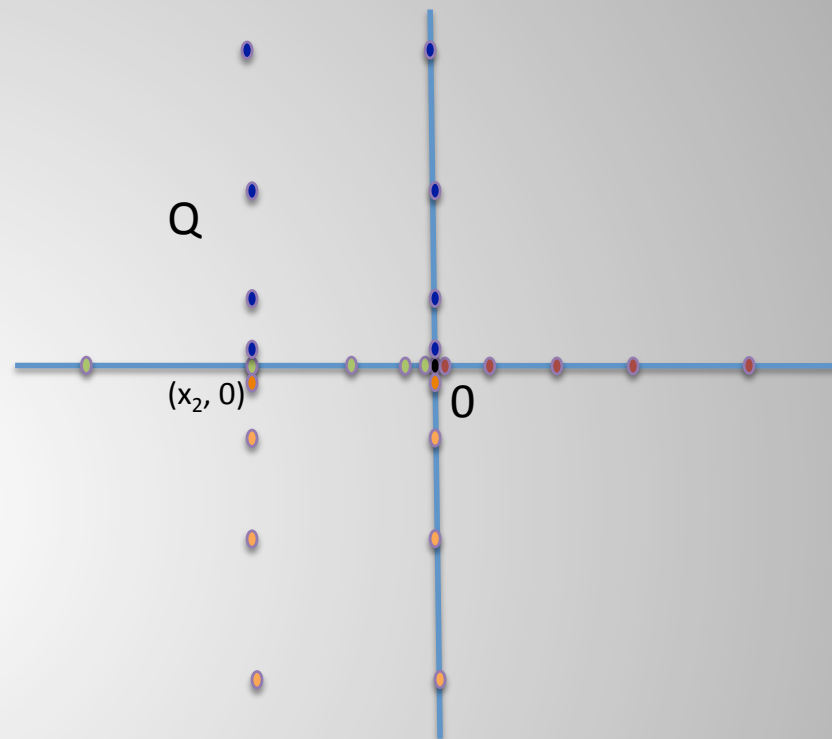
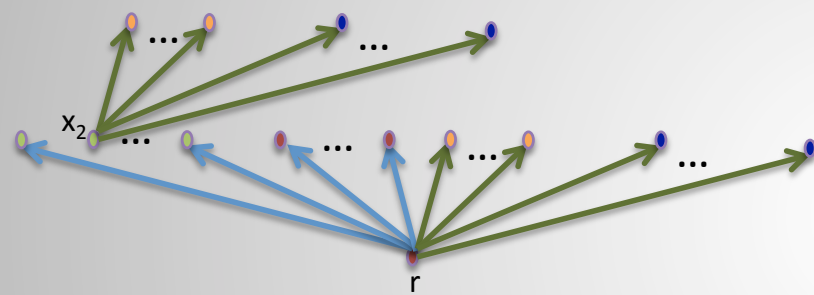
Definition 1 *A set $A \subseteq Q \times Q$ is horizontally open if with every $(x, y) \in A$ there exists an open interval $(x_1, x_2) \subseteq Q$ such that $x \in (x_1, x_2)$ and $(x_1, x_2) \times \{y\} \in A$*

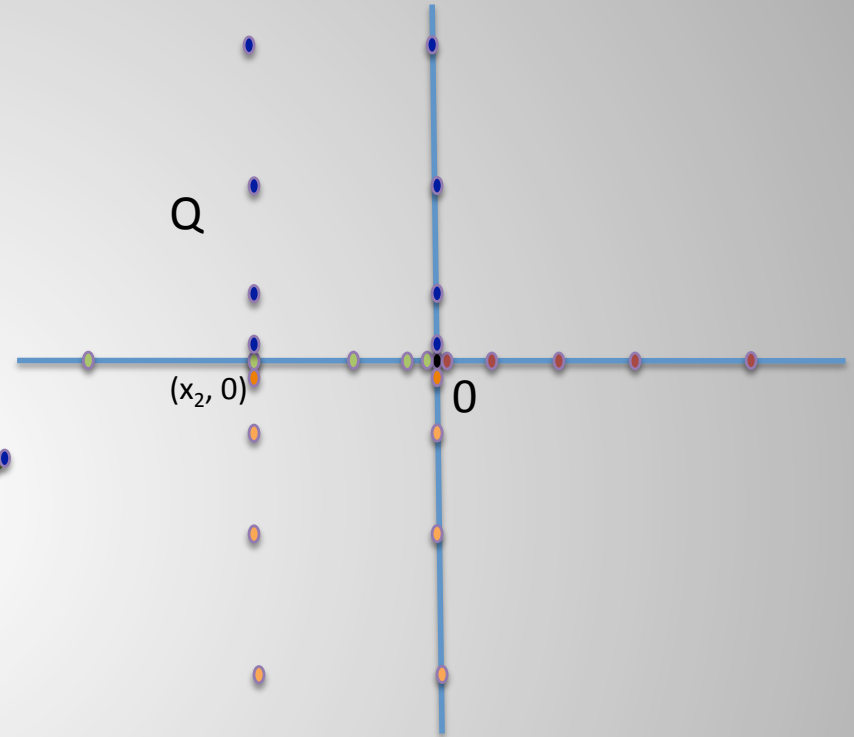
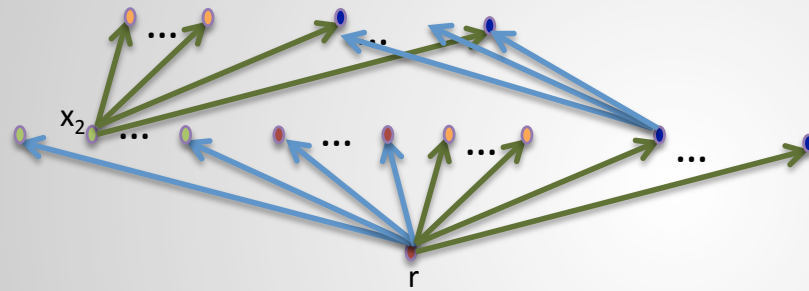
A set $A \subseteq Q \times Q$ is vertically open if with every $(x, y) \in A$ there exists an open interval $(y_1, y_2) \subseteq Q$ such that $y \in (y_1, y_2)$ and $\{x\} \times (y_1, y_2) \in A$



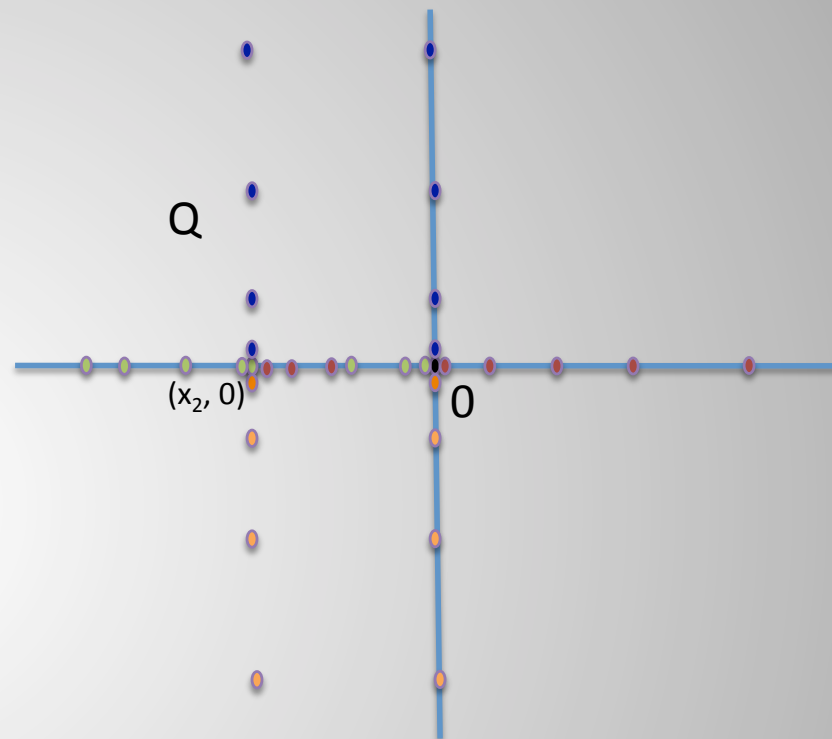
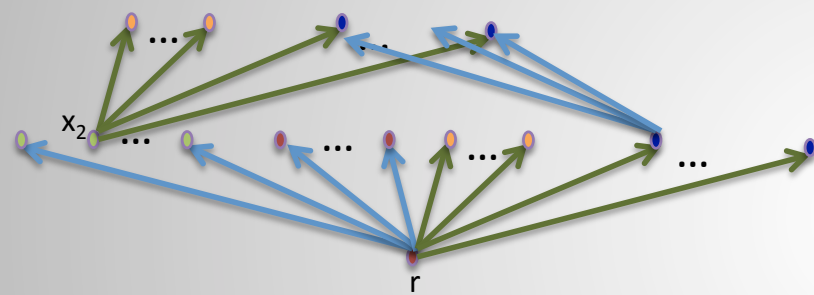


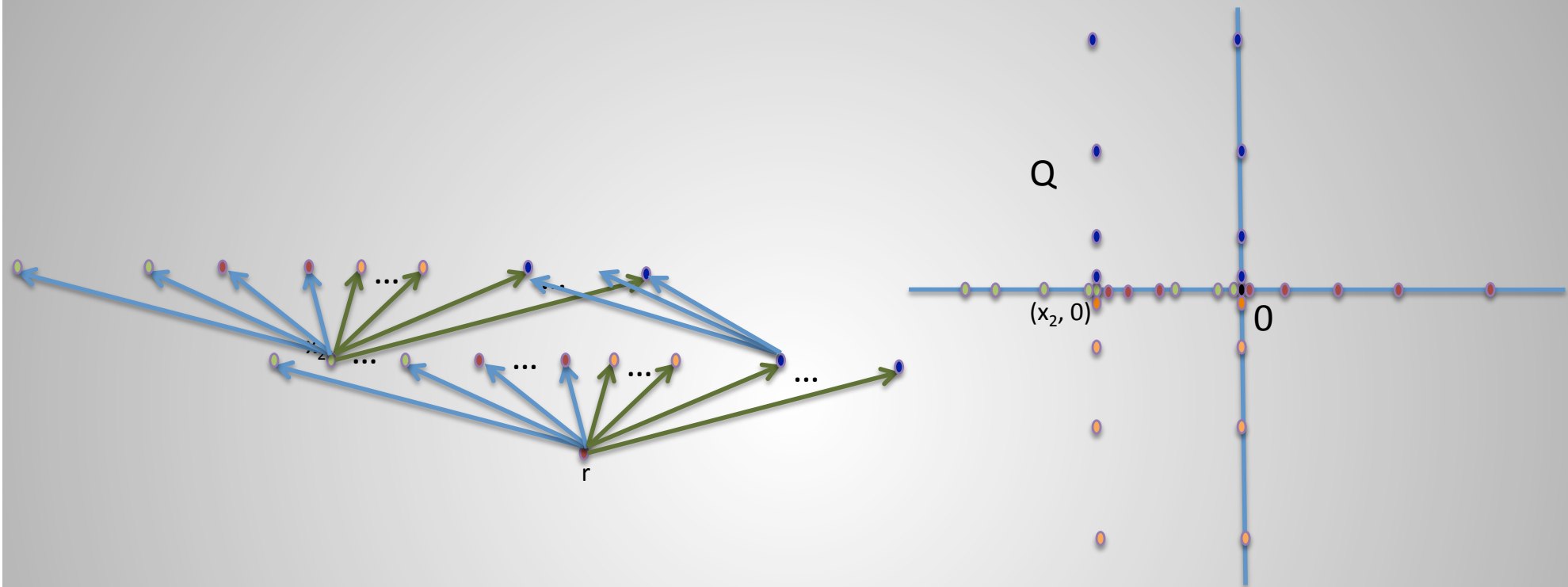






$$T_{\omega, \omega}^*$$





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d-morphism

DEFINITION 2.6. Suppose X is a topological space, $\mathfrak{F} = \langle W, R \rangle$ is a **wK4**-frame, and $f : X \rightarrow W$ is a map. We call f *irreflexively discrete*, or *i-discrete* for short, if $f^{-1}(w)$ is a discrete subspace of X for each irreflexive $w \in W$. We also call f *reflexively dense*, or *r-dense* for short, if $f^{-1}(w) \subseteq d_X f^{-1}(w)$ for each reflexive $w \in W$. Finally, we call f a *d-morphism* if (i) $f : X \rightarrow \overline{\mathfrak{F}}$ is interior, (ii) f is i-discrete, and (iii) f is r-dense.

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Fact 2 (Gabelaia) If f is a surjective d-morphism from (X, τ) onto (W, R) , then $(X, \tau) \Vdash \phi$ (in d-semantics) implies $(W, R) \Vdash \phi$ (in Kripke semantics).

Lemma 1. *There is a d -morphism from $T_{\omega,\omega}^*$ to $T_{n,n}$*

- Thank You!